Proposition 0.1. [16] Let \( u : B^n \rightarrow \Omega, \ n \geq 3 \) be twice differentiable q.c. mapping of the unit ball onto the bounded domain \( \Omega \) with \( C^2 \) boundary satisfying the differential inequality:
\[
|\Delta u| \leq A|\nabla u|^2 + B, \ A, B \geq 0.
\]
Then \( \nabla u \) is bounded and \( u \) is Lipschitz continuous.

Ovu prop nisam dokazivao; izgleda interesantno; U pristupu u ovom radu koji se verovatno može uopštititi lopta ima vaznu ulogu? O kojem radu mislite????

1. co-Lipschitz continuity

It is well-known that quasiconformal maps are locally well-behaved with respect to distance distortion. If \( f : \Omega \rightarrow \Omega' \) is a \( K \)-quasiconformal mapping between domains \( \Omega, \Omega' \subset \mathbb{R}^n \), then \( f \) is locally Hölder continuous with exponent \( \alpha = K^{1/(1-n)} \), i.e.
\[
|f(x) - f(y)| \leq M|x - y|^{\alpha}
\]
whenever \( x \) and \( y \) lie in a fixed compact set \( E \) in \( \Omega \). Here \( M \) is a constant depending only on \( K \) and \( E \) which can in general tend to infinity as the distance, from \( E \) to the boundary of \( \Omega \) tends to zero. However if the boundary of \( \Omega \) is enough "regular", then there hold an inequality similar to (1.1) uniformly in \( \Omega \) (see [10]). The following lemma in some form is proved in [19]. For the completeness we give its proof here and show that the constant is sharp.

Lemma 1.1. If \( u \in C^{1,1}_1 \) is a \( K \)-quasiconformal mapping, defined in a domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 3)\), then
\[
J_u(x) > 0, \ x \in \Omega
\]
providing that \( K < 2^{n-1} \). The constant \( 2^{n-1} \) is sharp.

Proof. Assume converse, i.e. \( J_u(a) = 0 \) for some \( a \in \Omega \). This implies \( \nabla u(a) = 0 \). Without loss of generality we can assume that, \( a = 0 \) and \( u(0) = 0 \). Let \( r < \text{dist}(0, \partial \Omega) \) and take \( E = B(0, r) \). Applying (1.1) to the mapping, \( f = u^{-1} \), defined in \( \Omega' = u(\Omega) \), we obtain
\[
|f(y)| \leq M_E|y|^K^{1/(1-n)} \text{ for } y \in u(E).
\]

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This implies

\[ M_E K^{1/(n-1)} |x|^{K^{1/(n-1)}} \leq |u(x)|, \text{ for } x \in E. \tag{1.3} \]

Now since \( u \) is twice differentiable, with \( \nabla u(0) = 0 \) and \( u(0) = 0 \), by Taylor formula, it follows that there exists a positive constant \( N \) such that

\[ |u(x)| \leq N|x|^2, \quad x \in E. \tag{1.4} \]

Combining (1.3) and (1.4) it follows that

\[ M_E^{-K^{1/(n-1)}} |x|^{K^{1/(n-1)}} \leq \frac{1}{N} |x|^2 - K^{1/(n-1)}, \quad x \in E. \tag{1.5} \]

This is only possible providing that

\[ 2 - K^{1/(n-1)} \leq 0. \]

And thus \( K \geq 2^{n-1} \) which is a contradiction.

To prove the sharpness of the result, take the mapping \( u(x) = |x|^\alpha x \), with \( \alpha \geq 1 \).

Then

\[ J_u(x) = (1 + \alpha)|x|^{\alpha}, \tag{1.6} \]

and

\[ |\nabla u(x)| = (\alpha + 1)|x|^\alpha. \tag{1.7} \]

By (1.6) and (1.7) it follows that

\[ \frac{|\nabla u(x)|^\alpha}{J_u(x)} = (\alpha + 1)^{n-1}. \]

Therefore \( u \) is twice differentiable \((1 + \alpha)^{n-1}\)-quasiconformal self-mapping of the unit ball with \( J_u(0) = 0 \). This means that the constant \( 2^{n-1} \) is the best possible.

**Lemma 1.2.** Let \( u \) be a harmonic mapping of the unit ball into itself and let \( u(0) = 0 \). Then there exists a constant \( C_n \) such that

\[ \frac{1 - |x|^2}{1 - |u(x)|^2} \leq C_n, \quad x \in B^n. \tag{1.8} \]

**Proof.** Let \( S^+ \) denotes the northern hemisphere and let \( S^- \) denotes the southern hemisphere. Let \( U = P[\chi_S^+] - P[\chi_S^-] \) be the poisson integral of a function that equals 1 on \( S^+ \) and \(-1\) on \( S^- \). Then by Schwartz lemma ([1]), for fixed \( x_0 \) there holds the inequality

\[ \left< u(x), \frac{u(x_0)}{|u(x_0)|} \right> \leq |U(|x|N)|, \]

where \( N \) is the north pole.

It follows that

\[ |u(x_0)|^2 \leq |U(|x_0|N)|^2. \]

Thus

\[ \frac{1 - |x|^2}{1 - |u(x)|^2} \leq \frac{1 - |x|^2}{1 - |U(|x|N)|^2} =: g(r), \quad r = |x|. \]
Now we need the following lemma.

**Lemma 1.3** (Hopf’s Boundary Point lemma). [18] and [5]. Let \( v \) satisfies \( \Delta v \geq 0 \) in \( D \) and \( v \leq M \) in \( D \), \( v(P) = M \) for some \( P \in \partial D \). Assume that \( P \) lies on the boundary of a ball \( B \subset D \). If \( v \) is continuous on \( D \cup P \) and if the outward directional derivative \( \frac{\partial v}{\partial n} \) exists at \( P \), then \( v \equiv M \) or \( \frac{\partial v}{\partial n} > 0 \).

Let apply this lemma to the function \( U(x) \) and take \( h(r) = U(rN) \). We obtain
\[
h'(1) = \frac{\partial U(N)}{\partial n} > 0.
\]
Thus
\[
C_n := \sup_{|x| \leq 1} \left\{ \frac{1 - |x|^2}{1 - U(|x|N)^2} \right\} < \infty.
\]
The results follows.

**□**

**Lemma 1.4.** If \( p \) is a Möbius transformation of the unit ball onto itself, then for every \( k, l \in \mathbb{N} \) there exist constants \( C_{k,l} \) such that
\[
(1.9) \quad \frac{k!|a|^{k-1}(1 - |a|^2)}{|x,a|^{k+1}} \leq \frac{|p^{(k)}(x)|}{|x,a|^{k+1}} \leq \frac{C_{k,0}|a|^{k-1}(1 - |a|^2)}{|x,a|^{k+1}}, \quad x \in B^n, \quad p(0) = a.
\]

\[
(1.10) \quad \frac{|p^{(k)}(x)|}{|p^{(l)}(x)|} \leq \frac{1}{C_{k,l}} \frac{1}{(1 - |x|)^{k-l}} \quad x \in B^n.
\]

and
\[
(1.11) \quad |p^{(k)}(x)| \leq \frac{C_{k,0}}{(1 - |p(0)|^2)^{(k-1)/2}} \left( \frac{1 - |p(x)|^2}{1 - |x|^2} \right)^{(k+1)/2} \quad x \in B^n.
\]

**Proof.** Since
\[
|\nabla p| = \frac{1 - |a|^2}{|x,a|^2},
\]
using the fact that
\[
|\nabla|\nabla p| \leq |\nabla p|,
\]
it follows that
\[
\frac{2|a|(1 - |a|^2)}{|x,a|^3} \leq |\nabla p|.
\]
The rest of the proof of left-hand side of (1.9) follows by induction.

Some of the formula which we use here are taken form the excellent book of Ahlfors [2]. From
\[
[x,a]^2 = |x|^2|x^* - a|^2 = |a|^2|x - a^*|^2
\]
and
\[
= |x|a - |x||x|^* = ||a|x - |a||a^*| = ||x|a - |x||x^*| \cdot ||a|x - |a||a^*|,
\]
it follows that
\[ [x, a]^2 \geq (1 - |a|)(1 - |x|). \]

Assume that \( p \) is not an identity and let \( p(0) = -a \) or \( p(a) = 0 \). Then
\[
p(x) = \frac{(1 - |a|^2)(x - a) - |x - a|^2a}{[x, a]^2},
\]
and
\[
P'(x) = \frac{1 - |a|^2}{[x, a]^2} \Delta(x, a),
\]
where
\[
\Delta(x, a) = (I - 2Q(a))(I - 2Q(x - a^*)),
\]
and \( Q(y) \) is the matrix which components have the form
\[
Q_{i,j}(y) = \frac{y_i y_j}{|y|^2}.
\]
For every \( y \) there holds \( K(y) := I - 2Q(y) \in O_n \) (is an orthogonal matrix). Thus \( \Delta(x, a) \) is an orthogonal matrix as well and consequently
\[
|\Delta(x, a)| = 1.
\]
This means that
\[
|p'(x)| = \frac{1 - |a|^2}{[x, a]^2}.
\]
Thus
\[
|p'(x)| \leq \frac{2}{1 - |x|}.
\]
If we put
\[
A = \frac{1 - |a|^2}{[x, a]^2}
\]
and
\[
B = (I - 2Q(a))(I - 2Q(x - a^*)),
\]
then we have that
\[
p' = AB,
\]
and consequently for \( k + 1 \)- derivative of \( p \) we have
\[
p^{(k+1)}(x) = \sum_{j=1}^{k} \binom{k}{j} A^{(j)} B^{(k-j)},
\]
treated as a \( k \) linear form between \( \mathbb{R}^n \times \cdots \times \mathbb{R}^n \) and \( M_{n \times n} \).

We will use the following notation
\[
Q(y) = \frac{y \otimes y}{|y|^2},
\]
where \( \otimes \) denotes the tensor product of vectors.

Differentiating we obtain
\begin{equation}
Q'(y)h_1 = \frac{h_1 \otimes y + y \otimes h_1}{|y|^2} - 2 \frac{\langle h_1, y \rangle y \otimes y}{|y|^4}
\end{equation}

(1.14)

Claim: For \( k \in \mathbb{N} \), there exists a \( 2k + 2 \) linear form \( P_k : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to M_{n \times n} \) such that

\begin{equation}
Q^{(k)}(y)(h_1, h_2, \ldots, h_k) = \frac{1}{|y|^{2k+2}} P^k(y, \ldots, y, h_1, \ldots, h_k).
\end{equation}

(1.15)

To prove this, we use the induction.

It is evident that, according to (1.14), this is true for \( k = 1 \).

Assume that (1.15) is true for \( k \) and prove it for \( k + 1 \). By (1.15) it follows that

\begin{equation}
Q^{(k+1)}(y)(h_1, h_2, \ldots, h_k, h_{k+1}) = \frac{1}{|y|^{2k+4}} \sum_{j=1}^{k+2} P^k(y, \ldots, y, \hat{h}_{k+1}, y, \ldots, h_1, \ldots, h_k)
\end{equation}

(1.16)

where \( \hat{h}_{k+1} \) denotes that \( h_{k+1} \) is in \( j \)th position.

Thus

\begin{equation}
Q^{(k+1)}(y)(h_1, h_2, \ldots, h_k, h_{k+1}) = \frac{P^{k+1}(y, \ldots, y, h_1, \ldots, h_k, h_{k+1})}{|y|^{2(k+1)+2}},
\end{equation}

(1.17)

where

\begin{equation}
P^{k+1}(e_1, \ldots, e_{k+3}, f_1, \ldots, f_{k+1})
= \sum_{j=1}^{k+2} < e_{k+3}, e_j > P^k(e_1, \ldots, f_{k+1}, \ldots, e_{k+2}, f_1, \ldots, f_k)
\end{equation}

(1.18)

Since \( P^k \) is an \( 2k + 2 \) linear form, it follows that

\begin{equation}
|P^k(y, \ldots, y, h_1, \ldots, h_k)| \leq |P^k||y|^{k+2} \prod_{j=1}^{k} |h_j|.
\end{equation}

(1.19)

Thus

\begin{equation}
|Q^k(y)| \leq \frac{|P^k|}{|y|^k},
\end{equation}

or what is the same
\begin{align}
|Q^k(x - a^*)| & \leq \frac{|P^k|}{|x - a^*|^k} = \frac{|a^k|P^k|}{|x, a|^k}.
\end{align}

To continue, observe that
\[B(x) = K(a)(I - 2Q(x - a^*)) , \quad K(a) \in O_n.\]

Thus
\[B^{(k)}(x) = -2K(a)Q^{(k)}(x - a^*),\]
and using the identity
\[1 - |p(x)|^2 = \frac{1 - |a|^2}{|x, a|^2}, \quad 1 - \frac{|a|^2}{|x - a^*|^2},\]
we obtain
\begin{align}
|B^k(x)| & \leq \frac{|a^k|P^k|}{|x, a|^k} < \frac{2|a^k|P^k|(1 - |p|^2)^{k/2}}{(1 - |a|^2)^{k/2}(1 - |x|^2)^{k/2}}. 
\end{align}

In order to estimate the derivatives of \(A(x) = \frac{1 - |a|^2}{|x, a|^2}\), define
\[
H(y) = \frac{1}{|y|^2} = \frac{|a|^2}{1 - |a|^2}A(x), \quad y = x - a^*.
\]

Similarly as above it can be proved that, for every \(k \geq 1\) there exists a \(2k\) linear form
\[G^k(x) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \mapsto \mathbb{R}\]
such that
\begin{align}
|H^{(k)}(y)(h_1, h_2, \ldots h_k) = \frac{1}{|y|^{2k+2}}G^k(y, \ldots , y, h_1, \ldots , h_k).
\end{align}

Therefore
\begin{align}
|H^k(y)| & \leq \frac{|G^k|}{|y|^{2k+2}},
\end{align}
and having in mind
\[1 - \frac{|p(x)|^2}{1 - |x|^2} = \frac{1 - |a|^2}{|x, a|^2},\]
it follows
\begin{align}
A^k(x) \leq \frac{(1 - |a|^2)|G^k|}{|a|^2|x - a^*|^{k+2}} = \frac{|a|^k(1 - |a|^2)|G^k|}{|x, a|^{k+2}}
& \leq \frac{2|a|^k|G^k|(1 - |p|^2)^{1+k/2}}{(1 - |a|^2)^{(k-1)/2}(1 - |x|^2)^{1+k/2}}.
\end{align}

Combining (1.13), (1.21) and (1.24) we obtain for \(k \geq 1:\)
\begin{align}
|p^{(k+1)}(x)| & \leq C_{k+1}\frac{1}{|x, a|^{k+1}},
\end{align}
and
\begin{align}
|p^{(k+1)}(x)| & \leq C_{k+1}\frac{1}{(1 - |x|)^{k+1}}.
\end{align}
where
\[ C_{k+1} = 2 \sum_{j=1}^{k} \binom{k}{j} |P^{j}| |G^{(k-j)}|. \]

This completes the proof. \(\square\)

**Remark 1.5.** In the plane case the sharp constant in (1.26) is \(C_k = 2k!\). The question arises, is this the best constant for arbitrary \(n\)? Notice that in (1.12) is showed that this is the case for \(k = 1\).

**Theorem 1.6.** If \(u\) is a q.c. harmonic mapping of the unit ball onto itself with \(K < 2^{n-1}\) then
\[ J_u(x) \geq c_K > 0 \quad \text{for} \quad x \in B^n. \]

**Proof.** We will use similar approach as in [19]. We will prove that the function \(|\nabla u|\) is uniformly bounded below away from 0 by contradiction. Suppose not, then there exists a sequence of points \(x_i \in B^n\), such that \(\nabla u(x_i) \to 0\) as \(i \to \infty\). To finish the proof we prove the following lemma:

**Lemma 1.7.** Let \(u\) be a harmonic Lipschitz mapping of the unit ball into itself. Let \(x_i\) be a sequence of points. Let \(p_i\) and \(q_i\) be two Möbius transformations of \(B^n\) such that \(q_i(0) = x_i\) and \(p_i(u(x_i)) = 0\). Take \(u_i = p_i \circ u \circ q_i\). Then
\[ |D^{(k)}u_i(x)| \leq C_n^k \frac{1}{(1 - |x|^2)^k}, \quad k \in \mathbb{N}, \]

where \(C_n^k\) is independent on \(x\) and \(i\).

**Proof.** In order to simplify calculations, sometimes along this proof, we will avoid the arguments of functions.

Using
\[ |\nabla p_i(u)| = \frac{1 - |p_i(u)|^2}{1 - |u|^2}, \]
and
\[ |\nabla q_i(x)| = \frac{1 - |q_i(x)|^2}{1 - |x|^2}, \]

it follows that
\[ |\nabla u_i| \leq |\nabla p_i||\nabla u||\nabla q_i| \]
\[ \leq \frac{1 - |p_i(u(q_i(x)))|^2}{1 - |u(q_i(x))|^2} \frac{1 - |q_i(x)|^2}{1 - |x|^2} |\nabla u| \]
\[ \leq C_n |\nabla u| \frac{1 - |p_i(u(q_i(x)))|^2}{1 - |x|^2}. \]

Thus
\[ |\nabla u_i| \leq C_n |\nabla u| \frac{1}{1 - |x|^2}. \]

For \(m \in \mathbb{N}\), we make use of the Cauchy inequalities:
\[ |D^m(u)(q_i(x))| \leq A_n |\nabla u|_\infty \frac{1}{(1 - |q_i(x)|)^{m-1}}. \]
To establish the behaviors of $D^ku_k$, $k > 1$, we use induction. It is evident that, $D^ku_k$ is complicate to compute for large $k$, however it is claere that it can be written as the following sum:

$$D^ku_k = \sum \left( p_1^{(\tau)} \prod_{i=1}^n D^{j_i}u_{q_i}^{(s_{i1})} \ldots \cdot q_i^{(s_{it_i})} \right),$$

where $\prod$ denotes corresponding product of linear operators. Here the index $\tau$ ranges from 1 to $k$, and the other indexes $j_i, s_{i1}, \ldots, s_{it_i}$ satisfy similar bounds.

Since

$$|p_1^{(\tau)}| \prod_{i=1}^n |D^{j_i}u_i| |q_i^{(s_{i1})}| \ldots |q_i^{(s_{it_i})}| \leq C |p_1^{(\tau)}| \prod_{i=1}^n \frac{\|
abla u_i\|_\infty}{1 - |q_i(x)|^{j_i-1}} |q_i^{(s_{i1})}| \ldots |q_i^{(s_{it_i})}|,$$

It is enough to prove that

$$|p_1^{(\tau)}| \prod_{i=1}^n \frac{\|
abla u_i\|_\infty}{1 - |q_i(x)|^{j_i-1}} |q_i^{(s_{i1})}| \ldots |q_i^{(s_{it_i})}| \leq C \frac{1}{(1 - |x|)^k}.$$

For $k = 1, (1.34)$ is true. Assume that (1.34) is true for $k$ and therefore (1.33) is true as well. In what follows we are going to prove (1.34) for $k + 1$

Since $D^{k+1}u_k = DD^ku_k$, it follows that, a corresponding formula (1.33) for $D^{k+1}u_k$ instead of

$$p_1^{(\tau)} D^{k+1}u_k^{(s_{i1})} \ldots q_i^{(s_{it_i})}$$

contains

$$\left(p_1^{(\tau+1)} D^{k+1}u_k^{(s_{i1})} \ldots q_i^{(s_{it_i})} + p_1^{(\tau)} D^{k+1}u_k^{(s_{i1})} \ldots q_i^{(s_{it_i})} + \ldots + p_1^{(\tau)} D^{k+1}u_k^{(s_{i1})} \ldots q_i^{(s_{it_i})}\right),$$

and consequently the corresponding formula (1.34), instead of

$$|p_1^{(\tau)}| \frac{\|
abla u_i\|_\infty}{1 - |q_i(x)|^{j_i-1}} |q_i^{(s_{i1})}| \ldots |q_i^{(s_{it_i})}|$$

contains

$$\left(|p_1^{(\tau+1)}| |Du_i| |q_i| |D^{k+1}u_i| |q_i^{(s_{i1})}| \ldots |q_i^{(s_{it_i})}| + |p_1^{(\tau)}| \frac{\|
abla u_i\|_\infty}{1 - |q_i(x)|^{j_i-1}} |q_i^{(s_{i1})}| \ldots |q_i^{(s_{it_i})}| + \ldots \right).$$

Applying now (1.29), we get

$$\frac{\|
abla u\|_\infty}{(1 - |q_i(x)|)^{j_i-1}} |q_i'| = \frac{\|
abla u\|_\infty}{(1 - |q_i(x)|)^{j_i-1}} \frac{1 - |q_i(x)|^2}{1 - |x|^2} \frac{2}{1 - |x|^2}.$$
Next, by applying (1.8) and (1.10) we obtain
\begin{equation}
|p_{i}^{(\tau+1)}Duq_{i}/|p_{i}^{(\tau)}| \leq \frac{C|p_{i}^{(\tau)}|}{1-|u(q_{i}(x))|/1-|x|} \leq C \frac{|p_{i}^{(\tau)}|}{1-|x|}.
\end{equation}

On the other hand, according to (1.10) we have
\begin{equation}
|q_{i}^{(j+1)}| \leq C \frac{|q_{i}^{(j)}|}{1-|x|}.
\end{equation}

By induction, (1.34) is true for \(k\). The last fact and equation (1.35), (1.36) and (1.37) imply that (1.34) is true for \(k+1\). Consequently
\begin{equation}
|D^{(k)}u_{i}(x)| \leq C_{k} \frac{1}{(1-|x|^{2})^{k}}, \quad k \in \mathbb{N}
\end{equation}

Taking the notations of the previous lemma, \(u_{i} = p_{i} \circ u \circ q_{i}\) is a \(C^{\infty}K\)-quasiconformal mapping of the unit ball onto itself satisfying the condition \(u_{i}(0) = 0\) and
\begin{equation}
|\nabla u_{i}(0)| = \frac{1-|x_{i}|^{2}}{1-|u(x_{i})|} \nabla u(x_{i}) \rightarrow 0
\end{equation}
as \(i \rightarrow \infty\). By [6] for example, a subsequence of \(u_{i}\), also denoted by \(u_{i}\), converges uniformly to a \(K\)-quasiconformal map \(u\) on the close unit ball \(B^{n}\). According to this lemma, \(u\) is in \(C^{\infty}(B^{n};B^{n})\) with \(u(0) = 0\) and from (1.39) \(\nabla(u)(0) = 0\). This obviously contradicts the statement of Lemma 1.1. Hence the proof of the proposition is completed.

\[\square\]

Remark 1.8. Using the formula
\begin{equation}
|\nabla q_{i}(x)| = \frac{1-|x_{i}|^{2}}{|x_{i}|^{2}+|x_{i}|^{2}}
\end{equation}

First of all we have
\begin{equation}
1-|p_{i}(u(q_{i}(x)))|^{2} = \frac{(1-|u(x_{i})|^{2})(1-|u(q_{i}(x))|^{2})}{|u(x_{i})|^{2}|u(q_{i}(x)) - u(x_{i})|^{2}}
\end{equation}

Next there holds
\begin{equation}
1-|u(q_{i}(x))|^{2} \leq \nabla u_{\infty} \frac{1+|u(q_{i}(x))|}{1+|q_{i}(x)|} (1-|q_{i}(x)|^{2}) \leq 2\nabla u_{\infty}(1-|q_{i}(x)|^{2}),
\end{equation}
\begin{equation}
(1-|q_{i}(x)|^{2}) = |\nabla q_{i}|(1-|x|^{2}),
\end{equation}
and
\begin{equation}
|u(x_{i})|^{2}|u(q_{i}(x)) - u(x_{i})|^{2} \geq (1-|u(x_{i})|^{2}).
\end{equation}

Using one more again (1.8) and combining (1.41), (1.42), (1.43) and (1.44) it follows that
\[ 1 - \frac{|p_i(u_{q_i}(x))|^2}{1 - |x_i|^2} \leq 2|\nabla u_\infty|_{\infty} \frac{(1 + |u(x_i)|)^2}{|x_i||x - x_i^*|} \leq \frac{8C_n|\nabla u_\infty|}{|x_i||x + x_i^*|}. \]

Assume that \( \lim_{i \to \infty} x_i = t \). Combining (1.30) and (1.45) we obtain that for \( x \in B^\alpha \setminus \{ x : |x + t| \leq \varepsilon \} \),

\[ |\nabla u_i| \leq \frac{16}{\varepsilon} C_n^2|\nabla u_\infty|^2, \quad i \geq i_0. \]

It follows that \( u_n \) is locally uniformly Lipschitz family of q.c. mappings with locally bounded derivative.

**Theorem 1.9.** Let \( K < 2^{n-1} \) and assume that \( u \) is a \( K \)-q.c. harmonic mapping the unit ball onto itself. Then it is a bi-Lipschitz mapping.

**Proof.** First of all

\[ 1/c < |\nabla u| \leq c, \]

then using the fact \( u \) is quasi-conformal, it follows that

\[ 1/c_1 < |\nabla (u^{-1})| \leq c_1 \]

and this implies that \( u \) is bi-Lipschitz. \( \square \)

In what follows is given an non-trivial example of q.c. harmonic selfmapping of the unit ball.

**Example 1.10.** Let \( I_\varepsilon(x) = (x_1 + \varepsilon, x_2, x_3) \) then

\[ |I_\varepsilon(x)| = (1 + 2\varepsilon x_1 + \varepsilon^2)^{1/2}. \]

Define

\[ \phi_\varepsilon(x) = \frac{I_\varepsilon(x)}{|I_\varepsilon(x)|} = (1 + 2\varepsilon x_1 + \varepsilon^2)^{-1/2}(x_1 + \varepsilon, x_2, x_3) \]

and take \( \Phi_\varepsilon = P[\phi_\varepsilon] \). Then for small enough \( \varepsilon \) \( \Phi_\varepsilon \) is a diffeomorphism of the unit ball onto itself having a diffeomorphic extension to the boundary. This for example means that \( \Phi_\varepsilon \) is q.c.

It is enough to prove that \( \Phi_\varepsilon \) is injective in \( \overline{B^\varepsilon} \) for small \( \varepsilon \). In order to do this we use the following result due to Gilbarg and Hörmander see [7, Theorem 6.1 and Lemma 2.1].

**Proposition 1.11.** The Dirichlet problem \( \Delta u = f \) in \( \Omega \), \( u = u_0 \) on \( \partial \Omega \in C^1 \) has a unique solution \( u \in C^{1,\alpha} \), for every \( f \in C^0, \alpha \), and \( u_0 \in C^{1,\alpha} \), and we have

\[ ||u||_{1,\alpha} \leq C(||u_0||_{1,\alpha,\partial \Omega} + ||f||_{0,\alpha}) \]

where \( C \) is a constant.
Direct calculations yield
\[ |\nabla \Phi_{\varepsilon}(x) - Id| = |\nabla P[\Phi_{\varepsilon} - Id](x)| \]
\[ \leq C \sup_{|x|=1} \left\{ \left( \sum_{i=1}^{3} |\partial_{x_i}(\nabla \Phi_{\varepsilon}(x) - Id)|^2 \right)^{1/2} + \left( \sum_{i=1, j=1}^{3} |\partial_{x_i, x_j}(\nabla \Phi_{\varepsilon}(x) - Id)|^2 \right)^{1/2} \right\} \]
\[ = C \times \sup_{|x|=1} \left\{ \left( -1 + \frac{\varepsilon x_1 + 1}{1 + \varepsilon^2 + 2\varepsilon x_1} \right)^2 + 2 \left( -1 + \frac{1}{\sqrt{1 + \varepsilon^2 + 2\varepsilon x_1}} \right)^2 \right. \]
\[ + \left. \left( \frac{\varepsilon^2 x_2^2}{1 + \varepsilon^2 + 2\varepsilon x_1} - \frac{\varepsilon x_3}{1 + \varepsilon^2 + 2\varepsilon x_1} \right)^{1/2} + \left( 2(-1 + \frac{1}{\sqrt{1 + a^2 + 2ax_1}})^2 \right) \right\}. \]

Therefore
\[ \lim_{\varepsilon \to 0} |\nabla \Phi_{\varepsilon}(x) - Id| = 0 \]
uniformly on \( B^n \).

It follows that there exist \( \varepsilon > 0 \) such that
\[ \sup_{|x|\leq 1} |\nabla \Phi_{\varepsilon}(x) - Id| < 1/2. \]

From
\[ |\Phi_{\varepsilon}(x) - \Phi_{\varepsilon}(y) + y - x| \leq 1/2|x - y|, \]
we obtain
\[ 1/2|x - y| \leq |\Phi_{\varepsilon}(x) - \Phi_{\varepsilon}(y)|. \]

This implies that, \( \Phi_{\varepsilon} \) is injective.

**Remark 1.12.** It seems that, the previous example can be modified to the class of all bi-Lipschitz harmonic diffeomorphisms of the unit ball onto itself. Thus small perturbations of the boundary value of harmonic q.c. transformation \( \phi \in C^2(B^n) \), of the unit ball onto itself induce harmonic q.c. mappings.

**References**


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