ON QUASICONFORMAL SELF-MAPPINGS OF THE UNIT DISK SATISFYING POISSON’S EQUATION

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Abstract. Let \( QC(K, g) \) be a family of \( K \) quasiconformal mappings of the open unit disk onto itself satisfying the PDE \( \Delta w = g, g \in C(U) \), \( w(0) = 0 \).

It is proved that \( QC(K, g) \) is a uniformly Lipschitz family. Moreover, if \( |g|_\infty \) is small enough, then the family is uniformly bi-Lipschitz. The estimations are asymptotically sharp as \( K \to 1 \) and \( |g|_\infty \to 0 \), so \( w \in QC(K, g) \) behaves almost like a rotation for sufficiently small \( K \) and \( |g|_\infty \).

1. Introduction and statement of the main result

In this paper \( U \) denotes the open unit disk in \( \mathbb{C} \), and \( S^1 \) denotes the unit circle.

Also, by \( D \) and \( \Omega \) we denote open regions in \( \mathbb{C} \). For a complex number \( z = x + iy \), its norm is given by \( |z| = \sqrt{x^2 + y^2} \). For a real \( 2 \times 2 \) matrix

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},
\]

we will consider the matrix norm \( |A| = \sup \{|Az| : |z| = 1\} \) and the matrix function \( l(A) = \inf \{|Az| : |z| = 1\} \).

A real-valued function \( u \), defined in an open subset \( D \) of the complex plane \( \mathbb{C} \), is harmonic if it satisfies Laplace’s equation:

\[
\Delta u(z) := \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 0 \quad (z = x + iy).
\]

A complex-valued function \( w = u + iv \) is harmonic if both \( u \) and \( v \) are real harmonic.

We say that a function \( u : D \to \mathbb{R} \) is ACL (absolutely continuous on lines) in the region \( D \), if for every closed rectangle \( R \subset D \) with sides parallel to the \( x \) and \( y \)-axes, \( u \) is absolutely continuous on a.e. horizontal and a.e. vertical line in \( R \). Such a function has of course, partial derivatives \( u_x, u_y \) a.e. in \( D \).

The definition carries over to complex valued functions.

Definition 1.1. A homeomorphism \( w : D \to \Omega \), between open regions \( D, G \subset \mathbb{C} \), is \( K \)-quasiconformal (\( K \geq 1 \)) (abbreviated \( K-q.c. \)) if

1. \( w \) is ACL in \( D \),
2. \( |w_z| \leq k|w_z| \) a.e. (\( k = \frac{K}{K+1} \)).

Here

\[
w_z := \frac{1}{2} (w_x - iw_y) \quad \text{and} \quad w_x := \frac{1}{2} (w_x + iw_y)
\]

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are complex partial derivatives (cf. [1], pp. 3, 23–24).

If by $\nabla w(z)$ we denote the formal derivative of $w = u + iv$ at $z$:

$$\nabla w = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

then the condition (2) of Definition 1.1 can be written as

$$K^{-1}(|\nabla w|^2) \leq J_w \leq K(l(\nabla w))^2 \quad \text{a.e. on } D,$$

where $J_w = \det(\nabla u)$ is the Jacobian of $w$. The above fact follows from the following well-known formulae

$$J_w(z) = |w_z|^2 - |w_\bar{z}|^2, \quad |\nabla w| = |w_z| + |w_\bar{z}|, \quad l(\nabla w) = ||w_z| - |w_\bar{z}||.$$

Notice that if $w$ is $K$–quasiconformal, then $w^{-1}$ is $K$–quasiconformal as well (this follows from (1.1)).

Let $P$ be the Poisson kernel, i.e. the function

$$P(z, e^{i\theta}) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2},$$

and let $G$ be the Green function of the unit disk, i.e. the function

$$G(z, \omega) = \frac{1}{2\pi} \log \left| \frac{1 - z\bar{\omega}}{z - \omega} \right| \quad z, \omega \in U, \ z \neq \omega.$$

The functions $z \mapsto P(z, e^{i\theta})$, $z \in U$, and $z \mapsto G(z, \omega)$, $z \in U \setminus \{\omega\}$ are harmonic.

Let $f : S^1 \to \mathbb{C}$ be a bounded integrable function on the unit circle $S^1$ and let $g : U \to \mathbb{C}$ be continuous. The solution of the equation $\Delta w = g$ in the unit disk satisfying the boundary condition $w|_{S^1} = f \in L^1(S^1)$ is given by

$$w(z) = P[f](z) - G[g](z)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta}) f(e^{i\theta}) d\theta - \int_{\partial U} G(z, \omega) g(\omega) \, dm(\omega),$$

$|z| < 1$, where $dm(\omega)$ denotes the Lebesgue measure in the plane. It is well known that if $f$ and $g$ are continuous in $S^1$ and in $\overline{U}$ respectively, then the mapping $w = P[f] - G[g]$ has a continuous extension $\tilde{w}$ to the boundary, and $\tilde{w} = f$ on $S^1$. See [9, pp. 118–120].

We will consider those solutions of the PDE $\Delta w = g$ that are quasiconformal as well and will investigate their Lipschitz character.

Recall that a mapping $w : D \to \Omega$ is said to be $C$–Lipschitz ($C > 0$) ($c$–co-Lipschitz ($c > 0$)) if

$$|w(z_1) - w(z_2)| \leq C|z_1 - z_2|, \quad z_1, z_2 \in D,$$

$$0 < c < \infty.$$
[28], [26], [30] for additional results concerning the Lipschitz character of harmonic quasiconformal mappings w.r.t the hyperbolic metric.

The following theorem is a generalization of an analogous theorem for the unit disk due to Pavlović [21] and of an asymptotically sharp version of Pavlović theorem due to Partyka and Sakan [20] in the case of harmonic quasiconformal mappings.

The following fact is the main result of the paper.

**Theorem 1.2.** Let \( K \geq 1 \) be arbitrary and let \( g \in C(\mathbb{U}) \) and \( |g|_\infty := \sup_{w \in \mathbb{U}} |g(w)| \).

Then there exist constants \( N(K) \) and \( M(K) \) with \( \lim_{K \to 1} M(K) = 1 \) such that:

\[
\left( \frac{1}{M(K)} - \frac{7 |g|_\infty}{6} \right) |z_1 - z_2| \leq |w(z_1) - w(z_2)| \leq (M(K) + N(K)|g|_\infty)|z_1 - z_2|.
\]

The proof of Theorem 1.2, given in Section 3, depends on the following two propositions:

**Proposition 1.3.** [13] Let \( w \) be a quasiconformal \( C^2 \) diffeomorphism from a bounded plane domain \( D \) with \( C^{1,\alpha} \) boundary onto a bounded plane domain \( \Omega \) with \( C^{2,\alpha} \) boundary. If there exist constants \( a \) and \( b \) such that

\[
(1.6) \quad |\Delta w| \leq a|\nabla w|^2 + b, \quad z \in D,
\]

then \( w \) has bounded partial derivatives in \( D \). In particular it is a Lipschitz mapping in \( D \).

**Proposition 1.4 (Mori’s Theorem).** [5, 22, 31] If \( w \) is a \( K \)-quasiconformal self-mapping of the unit disk \( U \) with \( w(0) = 0 \), then there exists a constant \( M_1(K) \), satisfying the condition \( M_1(K) \to 1 \) as \( K \to 1 \), such that

\[
(1.7) \quad |w(z_1) - w(z_2)| \leq M_1(K)|z_1 - z_2|^{K^{-1}}.
\]

See also [2] and [19] for some constants that are not asymptotically sharp.

The mapping \(|z|^{-\alpha} z^{-K^{-1}} z \) shows that the exponent \( K^{-1} \) is optimal in the class of arbitrary \( K \)-quasiconformal homeomorphisms.

## 2. Auxiliary results

In this section, we establish some lemmas needed in the proof of the main results.

**Lemma 2.1.** Let \( w \) be a harmonic function defined on the unit disk and assume that its derivative \( v = \nabla w \) is bounded on the unit disk (or equivalently, according to Rademacher’s theorem [7], let \( w \) be Lipschitz continuous). Then there exists a mapping \( A \in L^\infty(S^1) \) defined on the unit circle \( S^1 \) such that \( \nabla w(z) = P[A](z) \) and for almost every \( e^{i\theta} \in S^1 \) the relation

\[
(2.1) \quad \lim_{r \to 1} \nabla w(re^{i\theta}) = A(e^{i\theta})
\]

holds. Moreover the function \( f(e^{i\theta}) := w(e^{i\theta}) \) is differentiable almost everywhere in \([0,2\pi]\) and the formula

\[
A(e^{i\theta}) \cdot (ie^{i\theta}) = \frac{\partial}{\partial \theta} f(e^{i\theta})
\]

holds.
Proof. For the proof of the first statement of the lemma, see, for example, [3, Theorem 6.13 and Theorem 6.39].

Next, since
\[ |\frac{\partial}{\partial \theta} w(re^{i\theta})| = |r \nabla w(re^{i\theta}) \frac{\partial}{\partial \theta} e^{i\theta}| \leq |r \nabla w(re^{i\theta})| \cdot |\frac{\partial}{\partial \theta} e^{i\theta}| \]
\[ \leq \sup_{\theta} |A(e^{i\theta})| \cdot |\frac{\partial}{\partial \theta} e^{i\theta}|, \]
the Lebesgue Dominated Convergence Theorem yields
\[ f(e^{i\theta}) = \lim_{r \to 1} w(re^{i\theta}) = \lim_{r \to 1} \int_{\theta}^{\theta} \frac{\partial}{\partial \varphi} w(re^{i\varphi}) d\varphi + f(e^{i\theta_0}) = \int_{\theta}^{\theta} \frac{\partial}{\partial \varphi} e^{i\varphi} d\varphi + f(e^{i\theta_0}). \]

Differentiating in $\theta$ we get
\[ \frac{\partial}{\partial \theta} f(e^{i\theta}) = A(e^{i\theta}) \cdot \frac{\partial}{\partial \theta} e^{i\theta} = A(e^{i\theta})(ie^{i\theta}) \]
almost everywhere in $S^1$. □

Lemma 2.2. If $f(e^{it}) = e^{i\psi(t)}$, $\psi(2\pi) = \psi(0) + 2\pi$, is a diffeomorphism of the unit circle onto itself, then
\[ |f(e^{it}) - f(e^{is})| \leq |\psi' \cdot |e^{it} - e^{is}|, \]
where $|\psi'| = \max_{0 \leq \tau \leq 2\pi} \{|\psi'(\tau)| : 0 \leq \tau \leq 2\pi\} = \max_{0 \leq \tau \leq 2\pi} \{|\partial_r f(e^{i\tau})| : 0 \leq \tau \leq 2\pi\}$. 

Proof. Take the function
\[ h(t) = \frac{|f(e^{it}) - f(e^{is})|}{|e^{it} - e^{is}|}. \]
Then we have
\[ h(t) = \frac{\sin \frac{\psi(t) - \psi(s)}{2}}{\sin \frac{t - s}{2}}. \]
In order to estimate the maximum of the function $h$, we found out that the stationary points of it satisfy the equation
\[ \tan \frac{\psi(t) - \psi(s)}{2} = \tan \frac{t - s}{2} \cdot \psi'(t). \]
Substituting (2.4) to (2.3) we obtain
\[
\begin{equation}
(2.5) \quad h^2(t) = \frac{1 + \tan^2 \frac{\psi(t) - \psi(s)}{2}}{1 + \tan^2 \frac{\psi(t) - \psi(s)}{2}} \psi'^2(t).
\end{equation}
\]

Now since
\[
2\pi = \psi(2\pi) - \psi(0) = \int_0^{2\pi} \psi'(\tau) d\tau,
\]

it follows that \(|\psi'|_\infty \geq 1\). If \(|\psi'(t)| \leq 1\) then from (2.5) it follows \(|h(t)| \leq |\psi'|_\infty\). If \(|\psi'(t)| > 1\), then again employing (2.5) we obtain \(|h(t)| \leq |\psi'|_\infty\). This implies the lemma.

**Lemma 2.3.** If \(z \in U\), and
\[
I(z) = \frac{1}{2\pi} \int_U \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} \, dm(\omega),
\]
then
\[
(2.6) \quad \frac{1}{2} \leq I(z) \leq \frac{2}{3}.
\]

Both inequalities are sharp. Moreover the function \(z \mapsto I(z)\), is a radial function and decreasing for \(|z| \in [0,1]\).

**Proof.** For a fixed \(z\), we introduce the change of variables
\[
\frac{z - \omega}{1 - \bar{z}\omega} = \xi,
\]
or, what is the same,
\[
\omega = \frac{z - \xi}{1 - \bar{z}\xi}.
\]

Then
\[
I := \frac{1}{2\pi} \int_U \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} \, dm(\omega)
\]
\[
= \frac{1}{2\pi} \int_U \frac{1 - |\xi|^2}{|\xi| \cdot |1 - \bar{z}\xi|} \, dm(\xi)
\]
\[
= \frac{1}{2\pi} \int_U \frac{1 - |\xi|^2}{|\xi| \cdot |1 - \bar{z}\xi|^2} (1 - |z|^2)^2 \, dm(\xi)
\]
\[
= \frac{1}{2\pi} \int_U \frac{(1 - |\xi|^2)(1 - |z|^2)^2}{|\xi|^3 |1 - \bar{z}\xi|^2} \, dm(\xi).
\]

Since
\[
1 - \bar{z}\xi = 1 - z \frac{z - \xi}{1 - \bar{z}\xi}
\]
\[
= \frac{1 - |z|^2}{1 - \bar{z}\xi},
\]
we see that
\[ I = \frac{1}{2\pi} \int_{\mathcal{U}} \frac{(1 - |z|^2)(1 - |\xi|^2)}{|\xi| \cdot |1 - z\xi|^4} \, d\nu(\xi). \]

In the polar coordinates, we have
\[ I = (1 - |z|^2) \int_0^1 (1 - \rho^2) \, d\rho \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1 - z\rho e^{it}|^4}. \]

By Parseval’s formula (see [24, Theorem 10.22]), we get
\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1 - z\rho e^{it}|^4} = \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{n=0}^{\infty} (n + 1) (\bar{z}\rho)^n e^{nit} \right)^2 \, dt = \sum_{n=0}^{\infty} (n + 1)^2 |z|^{2n}. \]

whence
\[ I = (1 - |z|^2) \sum_{n=0}^{\infty} \frac{2(n + 1)^2}{(2n + 1)(2n + 3)} |z|^{2n}. \]

Now the desired inequality follows from the simple inequality
\[ \frac{1}{2} \leq c_n := \frac{2(n + 1)^2}{(2n + 1)(2n + 3)} \leq \frac{2}{3} \quad (n = 0, 1, 2, \ldots). \]

Setting $|z|^2 = r$, and $\varphi(r) = I(z)$, we obtain
\[ \varphi'(r) = \sum_{n=1}^{\infty} n(c_n - c_{n-1}) r^{n-1}. \]

Since $c_n \leq c_{n-1}$ it follows that $\varphi$ is decreasing, as desired. \hfill \Box

We need the following well-known propositions.

**Proposition 2.4.** [25] Let $X$ be an open subset of $\mathbb{R}$, and $\Omega$ be a measure space. Suppose that a function $F: X \times \Omega \to \mathbb{R}$ satisfies the following conditions:

1. $F(x, \omega)$ is a measurable function of $x$ and $\omega$ jointly, and is integrable over $\omega$, for almost all $x \in X$ held fixed.
2. For almost all $\omega \in \Omega$, $F(x, \omega)$ is an absolutely continuous function of $x$.
   (This guarantees that $\partial F(x, \omega)/\partial x$ exists almost everywhere).
3. $\partial F / \partial x$ is "locally integrable" – that is, for all compact intervals $[a, b]$ contained in $X$:
   \[ \int_a^b \left| \frac{\partial}{\partial x} F(x, \omega) \right| \, dx \, d\omega < \infty. \]

Then $\int_\Omega F(x, \omega) \, d\omega$ is an absolutely continuous function of $x$, and for almost every $x \in X$, its derivative exists and is given by
\[ \frac{d}{dx} \int_\Omega F(x, \omega) \, d\omega = \int_\Omega \frac{\partial}{\partial x} F(x, \omega) \, d\omega. \]
The following proposition is well-known as well.

**Proposition 2.5.** [29, p. 24–26] Let \( \rho \) be a bounded (absolutely) integrable function defined on a bounded domain \( \Omega \subset \mathbb{C} \). Then the potential type integral

\[
I(z) = \int_{\Omega} \frac{\rho(\omega) \, dm(\omega)}{|z - \omega|}
\]

belongs to the space \( C(\mathbb{C}) \).

**Lemma 2.6.** Let \( \rho \) be continuous on the closed unit disc \( \mathbb{U} \). Then the integral

\[
J(z) = \frac{1}{2\pi} \int_{\mathbb{U}} \log \left| \frac{1 - z\overline{\omega}}{z - \omega} \right| \rho(\omega) \, dm(\omega)
\]

belongs to the space \( C^1(\mathbb{U}) \). Moreover

\[
\nabla J(z) = \frac{1}{2\pi} \int_{\mathbb{U}} \nabla \log \left| \frac{1 - z\overline{\omega}}{z - \omega} \right| \rho(\omega) \, dm(\omega).
\]

**Proof.** Straightforward calculations yield

\[
(2.8) \quad \nabla z \frac{1}{2\pi} \log \left| \frac{1 - z\overline{\omega}}{z - \omega} \right| = \frac{1}{2\pi} \left( \frac{1}{\omega - z} - \frac{\overline{\omega}}{1 - z\overline{\omega}} \right),
\]

and consequently

\[
(2.9) \quad \left| \nabla z \frac{1}{2\pi} \log \left| \frac{1 - z\overline{\omega}}{z - \omega} \right| \right| = \frac{1}{2\pi} \frac{1 - |\omega|^2}{|z - \omega||z\overline{\omega} - 1|}, \quad z \neq \omega.
\]

(Here \( \nabla \varphi(z, \omega) \) denotes the gradient of the function \( \varphi \) treated as a function of \( z \)). Let \( \Omega = \mathbb{U} \), and let \( \mu \) be the Lebesgue measure of \( \mathbb{U} \).

According to Lemma 2.3, condition (2.7) of Proposition 2.4 is satisfied. Applying now Proposition 2.4, and relation (2.8) together with Proposition 2.5, we obtain the desired conclusion. \( \square \)

**Lemma 2.7.** If \( g \) is continuous on \( \overline{\mathbb{U}} \), then the mapping \( Gg \) has a bounded derivative, i.e. it is Lipschitz continuous and the inequalities

\[
(2.10) \quad |\partial Gg| \leq \frac{1}{3} |g|_{\infty},
\]

and

\[
(2.11) \quad |\bar{\partial} Gg| \leq \frac{1}{3} |g|_{\infty}
\]

hold on the unit disk. Moreover \( \nabla Gg \) has a continuous extension to the boundary, and for \( e^{i\theta} \in S^1 \) there hold

\[
(2.12) \quad \partial Gg(e^{i\theta}) = -\frac{e^{i\theta}}{4\pi} \int_{|e^{i\theta} - \omega|^2} \frac{1 - |\omega|^2}{|e^{i\theta} - \omega|^2} g(\omega) \, dm(\omega),
\]

and

\[
(2.13) \quad \bar{\partial} Gg(e^{i\theta}) = -\frac{e^{i\theta}}{4\pi} \int_{|e^{i\theta} - \omega|^2} \frac{1 - |\omega|^2}{|e^{i\theta} - \omega|^2} g(\omega) \, dm(\omega).
\]

Finally, for \( e^{i\theta} \in S^1 \)
(2.14) \[ |\partial G| \leq \frac{1}{4} |g|_\infty, \]

and

(2.15) \[ |\partial G| \leq \frac{1}{4} |g|_\infty. \]

**Proof.** First of all for \( z \neq \omega \) we have

\[
G_z(z,\omega) = \frac{1}{4\pi} \left( \frac{1}{\omega - z} - \frac{\bar{\omega}}{1 - z\bar{\omega}} \right) = \frac{1}{4\pi} \frac{(1 - |\omega|^2)}{(z - \omega)(\bar{z}\omega - 1)},
\]

and

\[
G_{\bar{z}}(z,\omega) = \frac{1}{4\pi} \frac{(1 - |\omega|^2)}{(\bar{z} - \omega)(\bar{\omega}\bar{z} - 1)}.
\]

By Lemma 2.6 the potential type integral

\[
\partial G[z](z) = \frac{1}{4\pi} \int_U \frac{1 - |\omega|^2}{|z - \omega||\bar{z}\omega - 1|} g(\omega) \, dm(\omega),
\]

exists and belongs to the space \( C(U) \).

According to Lemma 2.3 it follows that

\[ |\partial G| \leq \frac{1}{4} |g|_\infty \int_U \frac{1 - |\omega|^2}{|z - \omega||\bar{z}\omega - 1|} \, dm(\omega), \]

and

\[ |\partial G| \leq \frac{1}{3} |g|_\infty. \]

The inequality (2.10) is proved. Similarly we establish (2.11).

According to Lemma 2.5 it follows

(2.16) \[ \partial G[f](z) = \int_U G_z(z,\omega) g(\omega) \, dm(\omega). \]

Next we have

(2.17) \[ \lim_{z \to e^{i\theta},z \in D} G_z(z,\omega) = -\frac{1}{4\pi} \frac{e^{-i\theta}(1 - |\omega|^2)}{|e^{i\theta} - \omega|^2} \]

and

(2.18) \[ \lim_{z \to e^{i\theta},z \in D} G_{\bar{z}}(z,\omega) = -\frac{1}{4\pi} \frac{e^{i\theta}(1 - |\omega|^2)}{|e^{i\theta} - \omega|^2}. \]

In order to deduce (2.12) from the last two relations, we use the Vitali theorem (see [6, Theorem 26.C]):

Let \( X \) be a measure space with finite measure \( \mu \), and let \( h_n : X \to \mathbb{C} \) be a sequence of functions that is uniformly integrable, i.e. such that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \), independent of \( n \), satisfying

\[ \mu(E) < \delta \implies \int_E |h_n| \, d\mu < \varepsilon. \]
Now: if \( \lim_{n \to \infty} h_n(x) = h(x) \) a.e., then
\[
\lim_{n \to \infty} \int_X h_n \, d\mu = \int_X h \, d\mu.
\]
(\dagger)

In particular, if
\[
\sup_n \int_X |h_n|^p \, d\mu < \infty, \quad \text{for some } p > 1,
\]
then (\dagger) and (\ddagger) hold.

Hence, to prove (2.12), it suffices to prove that
\[
\sup_{z \in \Omega} \int_{\Omega} \left( \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} \right) g(\omega) \, dm(\omega) < \infty, \quad \text{for } p = 3/2.
\]

In order to prove this inequality, we proceed as in the case of Lemma 2.3. We obtain
\[
I_{p,g}(z) = \int_{\Omega} \left( \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} \right)^p g(\omega) \, dm(\omega)
\]
\[
\leq |g|_\infty^p \int_{\Omega} \left( \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} \right)^p \, dm(\omega)
\]
\[
= |g|_\infty^p \int_{\Omega} \frac{(1 - |z|^2)^{2-p}(1 - |\omega|^2)^p}{|\xi|^p |1 - \bar{z}\xi|^4} \, dm(\xi)
\]
\[
\leq |g|_\infty^{3/2}(1 - |z|^2)^{1/2} \int_{0}^{1} \rho^{-1/2}(1 - \rho^2)^{3/2} \int_{0}^{2\pi} |1 - \bar{z}\rho e^{i\varphi}|^{-4} \, d\varphi
\]
\[
\leq |g|_\infty^{3/2}(1 - |z|^2)^{1/2} \int_{0}^{1} \rho^{-1/2}(1 - \rho^2)^{3/2}(1 - |z|^3) \, d\rho.
\]

Now the desired result follows from the elementary inequality
\[
\int_{0}^{1} \rho^{-1/2}(1 - \rho^2)^{3/2}(1 - |z|^3) \, d\rho \leq C(1 - |z|^2)^{-1/2}.
\]

This proves (2.12). Similarly we prove (2.13). The inequalities (2.14) and (2.15) follow from (2.12) and (2.13) and Lemma 2.3.

A mapping \( w : D \to \Omega \) is proper if the preimage of every compact set in \( \Omega \) is compact in \( D \). In the case where \( D = \Omega = U \), the mapping \( w \) is proper if and only if \( |w(z)| \to 1 \) as \( |z| \to 1 \).

**Lemma 2.8** (The main lemma). Let \( w \) be a solution of the PDE \( \Delta w = g \) that maps the unit disk onto itself properly. Let in addition \( w \) be Lipschitz continuous. Then there exist for a.e. \( t = e^{i\theta} \in S^1 \):
\[
(2.19) \quad \nabla w(t) := \lim_{r \to 1^-} \nabla w(rt)
\]
and
\[
(2.20) \quad J_w(t) := \lim_{r \to 1^-} J_w(re^{i\theta}),
\]
and the following relation
\[ J_w(t) = \psi'(\theta) \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(t) - f(e^{i\varphi})|^2}{|t - e^{i\varphi}|^2} d\varphi \]

(2.21)

\[ + \psi'(\theta) \int_0^1 r \left( \frac{1}{2\pi} \int_0^{2\pi} P(re^{i\varphi}, t) \langle g(rt), f(t) \rangle d\varphi \right) dr, \]

holds. Here \( \psi \) is defined by

\[ e^{i\psi(\theta)} := f(e^{i\theta}) = w|_{S^1(e^{i\theta})}. \]

If \( w \) is biharmonic (\( \Delta \Delta w = 0 \)), then we have:

\[ J_w(t) = \psi'(\theta) \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(t) - f(e^{i\varphi})|^2}{|t - e^{i\varphi}|^2} d\varphi \]

(2.22)

\[ + \frac{\psi'(\theta)}{2} \int_0^1 \langle g(rt), f(t) \rangle dr, \quad t \in S^1. \]

For an arbitrary continuous \( g \) and \( |g|_{\infty} = \max_{|z| \leq 1} |g(z)| \) the inequality

\[ |J_w(t) - \psi'(\theta)\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(t) - f(e^{i\varphi})|^2}{|t - e^{i\varphi}|^2} d\varphi| \leq \frac{\psi'(\theta)|g|_{\infty}}{2}, \quad t \in S^1 \]

holds.

Proof. First of all, according to Lemma 2.7, \( G[g] \) has a bounded derivative, and there exists the function \( \nabla G[g](e^{i\theta}), e^{i\theta} \in S^1 \), which is continuous and satisfies the limit relation

\[ \lim_{z \to e^{i\theta}, z \in D} \nabla G[g](z) = \nabla G[g](e^{i\theta}). \]

Since \( w = P[f] - G[g] \) has bounded derivative, from Lemma 2.1, it follows that there exists

\[ \lim_{r \to 1-} \nabla P[f](re^{i\theta}) = \nabla P[f](e^{i\theta}). \]

Thus \( \lim_{r \to 1-} \nabla w(re^{i\theta}) = \nabla w(e^{i\theta}). \)

It follows that the mapping \( \chi : \chi(\theta) = f(e^{i\theta}) = f(t), \ t \in S^1 \), defines the outer normal vector field \( n_{\chi} \) almost everywhere in \( S^1 \) at the point \( \chi(\theta) = f(e^{i\theta}) = e^{i\psi(\theta)} = (\chi_1, \chi_2) \) by the formula:

\[ n_{\chi}(\chi(\theta)) = \psi'(\theta) \cdot f(e^{i\theta}). \]

Let \( \varpi(r, \theta) := w(re^{i\theta}) \). According to Lemma 2.1, we obtain:

\[ \lim_{r \to 1-} \varpi(r, \theta) = \chi'(\theta). \]

(2.25)

On the other hand, for almost every \( \theta \in S^1 \) we have

\[ \frac{\chi_j(\theta) - \varpi_j(r, \theta)}{1 - r} = \varpi(r, \theta, \rho_j, \theta) \]

where \( r < \rho_{j,r,\theta} < 1, \ j = 1, 2 \). Thus we have:
\[(2.26) \quad \lim_{r \to 1^-} \varpi_j(r, \theta) = \lim_{r \to 1^-} \frac{\chi_j(\theta) - \varpi_j(r, \theta)}{1 - r}, \quad j \in \{1, 2\}.\]

Denote by \(p\) polar coordinates, i.e. \(p(r, \theta) = re^{i\theta}\).

We derive
\[
\lim_{r \to 1^-} J_{wop}(r, \theta) = \lim_{r \to 1^-} \left( \frac{\chi - P[f]}{1 - r}, \psi'(\theta) \cdot f(e^{i\theta}) \right) + \Lambda
\]
\[(2.27) = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + r}{|e^{i\theta} - re^{i\varphi}|^2} \left\langle f(e^{i\theta}) - f(e^{i\varphi}), \psi'(\theta) \cdot f(e^{i\theta}) \right\rangle d\varphi + \Lambda
\]
\[(2.28) = \psi'(\theta) \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(e^{i\theta}) - f(e^{i\varphi})}{|e^{i\theta} - e^{i\varphi}|^2} \right|^2 d\varphi + \Lambda,
\]
where
\[\Lambda = \lim_{r \to 1^-} \left( \frac{G|g|}{1 - r}, -i\chi_\theta \right).
\]

In order to estimate \(\Lambda\), observe first that
\[(2.29) \quad \lim_{z \to e^{i\theta}, \omega \in \mathbb{D}} \frac{G(z, \omega)}{1 - |z|} = \lim_{z \to e^{i\theta}, \omega \in \mathbb{D}} \frac{G(z, \omega) - G(e^{i\theta}, \omega)}{1 - |z|} = \frac{\partial G(re^{i\varphi}, \omega)}{\partial r}\bigg|_{r=1}.
\]

Since
\[\frac{\partial G(re^{i\varphi}, \omega)}{\partial r} = z_r G_z(re^{i\varphi}, \omega) + \bar{z}_r G_{\bar{z}}(re^{i\varphi}, \omega), \quad z_r = e^{i\theta}, \quad \bar{z}_r = e^{-i\theta},
\]
using (2.17) and (2.18) we obtain
\[(2.30) \quad \lim_{z \to e^{i\theta}, \omega \in \mathbb{D}} \frac{G(z, \omega)}{1 - |z|} = \frac{1}{2\pi} P(e^{i\theta}, \omega).
\]

On the other hand we have
\[(2.31) \quad J_{wop}(r, \theta) = rJ_w(re^{i\theta}).
\]

Combining (2.27), (2.29), (2.30) and (2.31) we obtain (2.21). Relations (2.22) and (2.23) follow from (2.21) and (1.3). If \(w\) is biharmonic, then \(g\) is harmonic and thus
\[(2.32) \quad \frac{1}{2\pi} \int_0^{2\pi} P(re^{i\varphi}, e^{i\theta}) \left\langle g(re^{i\varphi}), f(e^{i\theta}) \right\rangle d\varphi = \left\langle g(r^2 e^{i\theta}), f(e^{i\theta}) \right\rangle.
\]

This yields relation (2.22). \(\square\)
Lemma 2.9. If \( x \geq 0 \) is a solution of the inequality \( x \leq ax^a + b \), where \( a \geq 1 \) and \( 0 \leq aa < 1 \), then

\[
(2.32) \quad x \leq \frac{a + b - aa}{1 - aa}.
\]

Observe that for \( a = 0 \), (2.32) coincides with \( x \leq a + b \), i.e. \( x \leq ax^a + b \).

Proof. We will use the Bernoulli’s inequality. \( x \leq ax^a + b = a(x + 1)^a + b \leq a(1 + \alpha(x - 1)) + b \). Relation (2.32) now easily follows. \( \square \)

3. The main results

Theorem 3.1. Let \( g \in C(\overline{U}) \). The family \( QC(K, g) \) of \( K \)-quasiconformal \( (K \geq 1) \) self-mappings of the unit disk \( U \) satisfying the PDE \( \Delta w = g, w(0) = 0 \), is uniformly Lipschitz, i.e. there is a constant \( M' = M'(K, g) \) satisfying:

\[
(3.1) \quad |w(z_1) - w(z_2)| \leq M'|z_1 - z_2|, \quad z_1, z_2 \in U, \quad w \in QC(K, g).
\]

Moreover \( M'(K, g) \to 1 \) as \( K \to 1 \) and \( |g|_\infty \to 0 \).

In Remark 3.7 below is given a quantitative bound of \( M'(K, g) \).

Proof. Combining Proposition 1.3 and Lemma 2.8, in the special case where the range of a function is the unit disk, we obtain that there exist \( \nabla w \) and \( J_w \) almost everywhere in \( S^1 \), and the following inequality

\[
(3.2) \quad J_w(t) \leq \psi'(\theta) \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\varphi}) - f(e^{i\theta})|^2}{|e^{i\varphi} - e^{i\theta}|^2} d\varphi + \frac{|g|_\infty}{2} \right)
\]

holds.

Now from

\[
|\nabla w(re^{i\theta})|^2 \leq KJ_w(re^{i\theta}),
\]

we obtain

\[
(3.3) \quad \lim_{r \to 1-} |\nabla w(re^{i\theta})|^2 \leq \lim_{r \to 1-} KJ_w(re^{i\theta}),
\]

almost everywhere in \([0, 2\pi]\). From Lemma 2.1, we deduce that

\[
(3.4) \quad \lim_{r \to 1-} \frac{\partial(w(re^{i\theta}))}{\partial \theta} = \frac{\partial f(e^{i\theta})}{\partial \theta} = \psi'(\theta)e^{i\psi(\theta)}
\]

almost everywhere in \([0, 2\pi]\). Since

\[
\frac{\partial w \circ S}{\partial \theta}(r, \theta) = ru'(re^{i\theta})(ie^{i\theta}),
\]

using (3.4) we obtain that

\[
(3.5) \quad \psi'(\theta) \leq \lim_{r \to 1} |\nabla w(re^{i\theta})|.
\]

From (3.2)-(3.5) we infer that

\[
|\nabla w(e^{i\theta})|^2 \leq K|\nabla w(e^{i\theta})| \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\varphi}) - f(e^{i\theta})|^2}{|e^{i\varphi} - e^{i\theta}|^2} d\varphi + \frac{|g|_\infty}{2} \right)
\]

i.e.

\[
(3.6) \quad |\nabla w(e^{i\theta})| \leq K \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\varphi}) - f(e^{i\theta})|^2}{|e^{i\varphi} - e^{i\theta}|^2} d\varphi + \frac{|g|_\infty}{2} \right).
\]
Let

\[ M = \text{ess sup}_{0 \leq \tau \leq 2\pi} |\nabla w(e^{i\tau})|. \]

According to Lemma 2.2 and to relation (3.5) we obtain

\[ |f(e^{i\psi}) - f(e^{i\theta})| \leq M|e^{i\psi} - e^{i\theta}|. \]

Let \( \mu = K^{-1}, \gamma = -1 + K^{-2}, \) and let \( \nu = 1 - 1/K. \) Let \( \varepsilon > 0. \) Then there exists \( \theta_{\varepsilon} \) such that

\[ M(1 - \varepsilon) \leq |\nabla w(e^{i\theta_{\varepsilon}})|. \]

Applying now relation (3.6) and using (1.7), we obtain

\[ (1 - \varepsilon)M \leq K\left(M^\nu \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta_{\varepsilon}} - e^{i\varphi}| |f(e^{i\theta_{\varepsilon}}) - f(e^{i\varphi})|^{2-\nu} d\varphi + \frac{|g|_\infty}{2}\right)\]

\[ \leq KM^\nu M_1(K)^{1+\mu} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta_{\varepsilon}} - e^{i\varphi}|^{\gamma} d\varphi + K|g|_\infty \]

\[ \leq M_2(K)M^\nu + \frac{K|g|_\infty}{2}, \]

where

\[ M_2(K) = KM_1(K)^{1+\mu} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta_{\varepsilon}} - e^{i\varphi}|^{\gamma} d\varphi. \]

Letting \( \varepsilon \to 0 \) we obtain

\[ M \leq M_2(K)M^\nu + \frac{K|g|_\infty}{2}. \]

From (3.8) we obtain

\[ M \leq C_0 := \left(M_2(K) + \frac{K|g|_\infty}{2}\right)^{1/(1-\nu)} = \left(M_2(K) + \frac{K|g|_\infty}{2}\right)^{K}. \]

From Lemma 2.9, if

\[ M_1(K)^{1+\mu} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta_{\varepsilon}} - e^{i\varphi}|^{\gamma} d\varphi < \frac{1}{K-1} \]

and \( g \neq 0, \) we obtain

\[ M \leq C_1 := \frac{M_2(K) + K|g|_\infty/2 - \nu M_2(K)}{1 - \nu M_2(K)}. \]

Let \( C_2 := \min\{C_0, C_1\}. \)

If \( g \equiv 0 \) then by (3.9) we get

\[ M \leq C_2 := \left(M_2(K)\right)^{1/(1-\nu)}. \]

To continue observe that \( w - G[g] \) is harmonic. Thus

\[ |\nabla w(z) - \nabla G(z)| \leq \text{ess sup}_{0 \leq \tau \leq 2\pi} |\nabla w(e^{i\tau}) - \nabla G[g(e^{i\tau})]|. \]

According to Lemma 2.3 and Lemma 2.7 it follows that:

\[ |\nabla w(z)| \leq \text{ess sup}_{0 \leq \tau \leq 2\pi} |\nabla w(e^{i\tau})| + \frac{2}{3}|g|_\infty + \frac{1}{2}|g|_\infty. \]
Therefore the inequality (3.1) does hold for
\begin{equation}
M' = C_2 + \frac{7}{6} |g|_\infty.
\end{equation}

Using (1.7), it follows that
\[ \lim_{|g|_\infty \to 0, K \to 1} M'(K) = 1. \]

\[ \square \]

**Lemma 3.2.** If \( w \) is a \( K \)-q.c. self-mapping of the unit disk satisfying the PDE \( \Delta w = g \) and \( w(0) = 0 \), \( w|_{S^1(e^{i\theta})} = f(e^{i\theta}) = e^{i\psi(\theta)} \), \( g \in C(\overline{U}) \), then for almost every \( \theta \in [0, 2\pi] \) the relation
\begin{equation}
\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta}) - f(e^{i\varphi})|^2 \frac{d\varphi}{|e^{i\theta} - e^{i\varphi}|^2} \leq K \psi'(\theta) + \frac{|g|_\infty}{2}
\end{equation}
holds.

**Proof.** From (2.23) it follows that
\begin{equation}
\frac{J_w(e^{i\theta})}{\psi'(\theta)} = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta}) - f(e^{i\varphi})|^2 \frac{d\varphi}{|e^{i\theta} - e^{i\varphi}|^2} - \frac{|g|_\infty}{2}.
\end{equation}

Using Lemma 2.1 we obtain
\begin{equation}
\psi'(\theta) = \left| \frac{\partial f(e^{i\theta})}{\partial \theta} \right| = \left| \lim_{r \to 1^-} \frac{\partial w(re^{i\theta})}{\partial \theta} \right|.
\end{equation}

On the other hand
\begin{equation}
\frac{\partial w(re^{i\theta})}{\partial \theta} = izw_z(re^{i\theta}) - i\bar{z}w_{\bar{z}}(re^{i\theta}) \quad (z = re^{i\theta}).
\end{equation}

Therefore
\begin{equation}
\left| \lim_{r \to 1^-} \frac{\partial w(re^{i\theta})}{\partial \theta} \right| \geq \|w_z(t)| - |w_{\bar{z}}(t)|| = l(\nabla w(t)) \quad (t = e^{i\theta}).
\end{equation}

As \( w \) is \( K \)-q.c., according to (1.1) it follows that
\begin{equation}
\frac{J_w(t)}{(l(\nabla w(t)))^2} \leq K.
\end{equation}
Combining (3.14) - (3.18) we obtain (3.13). \[ \square \]

**Lemma 3.3.** Under the conditions and notations of Lemma 3.2, there exists a function \( m_1(K) \) such that \( \lim_{K \to 1} m_1(K) = 1 \) and
\begin{equation}
m(K) := \max \left\{ m_1(K) - \frac{4 - 5|g|_\infty}{4}, \frac{4 - 5|g|_\infty}{8} \right\} \leq K \psi'(\theta), \text{ for a.e. } \theta \in [0, 2\pi].
\end{equation}

**Proof.** Applying (1.7) to the mapping \( w^{-1} \), we obtain
\[ |f(z) - f(w)| \geq M_1(K)^{-K}|z_1 - z_2|^K. \]
Using now relation (3.13) we obtain
\[ K \psi'(\theta) \geq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} \, d\varphi - \frac{|g|_\infty}{2} \]

(3.20)

\[ \geq M_1(K)^{-2K} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - e^{i\varphi}|^{2K-2} \, d\varphi - \frac{|g|_\infty}{2} \]

\[ = m_1(K) - \frac{|g|_\infty}{2}, \]

where

\[ m_1(K) = M_1(K)^{-2K} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - e^{i\varphi}|^{2K-2} \, d\varphi. \]

Let us prove the second part of the inequality (3.19). Since \( w(0) = 0 \) we infer that \( P[f](0) = -G[g](0) \). Thus

\[ P[f](0) = \int_U G(0, \omega) g(\omega) \, dm(\omega), \]

i.e. in polar coordinates

\[ P[f](0) = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} r \log \frac{1}{r} g(\omega) \, dm(\omega). \]

Hence

\[ |P[f](0)| \leq |g|_\infty \int_0^1 \frac{1}{r} \log \frac{1}{r} \, dr = \frac{|g|_\infty}{4}. \]

Next we have

\[ K \psi'(\theta) \geq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} \, d\varphi - \frac{|g|_\infty}{2} \]

(3.21)

\[ \geq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \left( 1 - \text{Re} \left( \frac{f(e^{i\theta})}{e^{i\theta}} \right) \right) \, d\varphi - \frac{|g|_\infty}{2} \]

\[ \geq \frac{1 - |P[f](0)|}{2} - \frac{|g|_\infty}{2} \]

\[ \geq \frac{4 - 5|g|_\infty}{8}. \]

Combining (3.20) and (3.21) we obtain (3.19).

\[ \square \]

**Theorem 3.4.** If \( w \) is a \( K \)-q.c. orientation preserving self-mapping of the unit disk satisfying the PDE \( \Delta w = g \), \( w(0) = 0 \), \( g \in C(\mathbb{D}) \), then for

\[ m(K) = \max \left\{ M_1(K)^{-2K} \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - e^{i\varphi}|^{2K-2} \, d\varphi - \frac{|g|_\infty}{4}, \frac{4 - 5|g|_\infty}{8} \right\}, \]

the inequality

(3.22)

\[ l(\nabla w) \geq \frac{m(K)}{K^2} - \frac{7|g|_\infty}{6} \]

where \( l(\nabla w(z)) = \min \{|\nabla w(z)t| : |t| = 1\} \), holds.
Proof. Assume, as we may, that

\[
\frac{m(K)}{K^2} - \frac{|g|_\infty}{2} \geq \left( \frac{m(K)}{K^2} - \frac{7|g|_\infty}{6} \right) \geq 0.
\]

From (3.19) and the definition of quasiconformality we deduce that:

\[
\frac{m(K)}{K^2} \leq \psi'(\theta) \leq \frac{|\nabla w(e^{i\theta})|}{K} \leq l(\nabla w),
\]

i.e

\[
\frac{m(K)}{K^2} \leq |w_z| - |w_{\bar{z}}|
\]

almost everywhere on the unit circle.

According to relations (2.14) and (2.15) we obtain

\[
(3.24) \quad \frac{m(K)}{K^2} - \frac{|g|_\infty}{2} \leq |P[f]_z| - |P[f']_z|
\]

almost everywhere on the unit circle.

To continue observe that, as \( w \) is q.c., it follows that \( f \) is a homeomorphism. Hence by Choquet-Radó-Kneser theorem \( P[f] \) is a diffeomorphism (see [15], [4] or [23]).

Thus \( h := P[f] \) is a harmonic diffeomorphism. According to the Heinz theorem ([8])

\[
|h_z| + |h_{\bar{z}}| \geq \frac{1}{\pi^2},
\]

which, in view of the fact that \( |h_z| \geq |h_{\bar{z}}| \), implies that

\[
|h_z| \geq \frac{\sqrt{2}}{2\pi}.
\]

Thus the functions

\[
a(z) := \frac{h_{\bar{z}}}{h_z} \quad \text{and} \quad b(z) := \frac{1}{h_z}\left( \frac{m(K)}{K^2} - \frac{|g|_\infty}{2} \right)
\]

are holomorphic and bounded on the unit disk. As \( |a| + |b| \) is bounded on the unit circle by 1 (see (3.23) and (3.24)), it follows that it is bounded on the whole unit disk by 1 because

\[
|a(z)| + |b(z)| \leq P[|a|_z](z) + P[|b|_z](z), \quad z \in \mathbb{U}.
\]

This in turn implies that for every \( z \in \mathbb{U} \)

\[
(3.25) \quad l(\nabla h) \geq \frac{m(K)}{K^2} - \frac{|g|_\infty}{2}.
\]

By (2.10) and (2.11) we obtain

\[
(3.26) \quad l(\nabla w) \geq \frac{m(K)}{K^2} - \frac{1}{2}|g|_\infty - \frac{2}{3}|g|_\infty.
\]

Having in mind the fact \( l(\nabla w(z)) = |\nabla w^{-1}(w(z))|^{-1} \), and putting Theorem 3.1 and Theorem 3.4 together we obtain:
Theorem 3.5. Let $\mathcal{QC}(K,g)$ be the family of orientation preserving $K$-q.c. self-mappings of the unit disk satisfying the equation $\Delta w = g$ and $w(0) = 0$. Then for $|g|_\infty$ small enough (for example if $|g|_\infty \leq \frac{12}{15 + 28K^2}$) the family $\mathcal{QC}(K,g)$ is uniformly bi-Lipschitz, i.e. there exists $M_0(K,g) \geq 1$ such that

$$M_0(K,g)^{-1} \leq \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|} \leq M_0(K,g), \ w \in \mathcal{QC}(K,g), \ \text{for } z_1, z_2 \in \mathbb{U}, \ z_1 \neq z_2.$$ 

Moreover

$$\lim_{|g|_\infty \to 0, K \to 1} M_0(K,g) = 1.$$

Example 3.6. Let $w(z) = |z|^\alpha z$, with $\alpha > 1$. Then $w$ is twice differentiable $(1 + \alpha)$–quasiconformal self-mapping of the unit disk. Moreover

$$\Delta w = \alpha(2 + \alpha) \frac{|z|^\alpha}{z} = g.$$ 

Thus $g = \Delta w$ is continuous and bounded by $\alpha(2 + \alpha)$. However $w$ is not co-Lipschitz (i.e. it does not satisfy (1.5)), because $l(\nabla w)(0) = |w_z(0)| - |w_z(0)| = 0$. This means that the condition “$|g|_\infty$ is small enough” in Theorem 3.5 cannot be replaced by the condition “$g$ is arbitrary”.

Remark 3.7. Let $\mathcal{QC}_K(\mathbb{U})$ be the family of $K$–quasiconformal self-mappings of the unit disk. Let $M_1(K)$ be the Mori’s constant:

$$M_1(K) = \inf \{ M : |f(z_1) - f(z_2)| \leq M|z_1 - z_2|^{1/K}, z_1, z_2 \in \mathbb{U}, f \in \mathcal{QC}_K(\mathbb{U}), f(0) = 0 \}.$$ 

In [22] is proved that

$$M_1(K) \leq 16^{1 - 1/K} \min \left\{ \left( \frac{23}{8} \right)^{1-1/K}, (1 + 2^{3 - 2K})^{1/K} \right\}.$$ 

Since for $\alpha > -1$

$$\frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - e^{i\varphi}|^\alpha d\varphi = \frac{2^{\alpha+1}}{\pi} \frac{\sqrt{\pi \Gamma(\frac{1+\alpha}{2})}}{\alpha \Gamma(\frac{\alpha}{2})},$$

our proofs, in the case of harmonic mappings ($g \equiv 0$), yield the following estimates for co-Lipschitz constant

$$m_2 := \frac{2^{2K-2}\pi |K - 1/2|}{\sqrt{\pi(K^3 - K^2)\Gamma|K - 1|}M_1(K)^{2K}}$$

which is

$$\geq \frac{1}{K^2 M_1(K)^{2K}} \geq \frac{46^2}{K^2 46^{2K}}$$

and therefore is better than the corresponding constant

$$m_1 := \frac{2^{K(1-K^2)(3+1/K)/2}}{K^{3K+1}(K^2 + K - 1)^{3K}}$$

obtained in the paper [20] for every $K$ (see the appendix below).

Similarly we obtain the following estimate for the Lipschitz constant (see (3.9) and (3.12)).

$$M' = \left( KM_1(K)^{1+1/K} \left( \frac{2^{K-2}\pi |K - 2/2|}{\sqrt{\pi(K-2)^2 - 1)\Gamma|K - 2/2|}} + \frac{K|g|_\infty^2}{2} \right)^{1/K} + \frac{7|g|_\infty}{6}. \right.$$


The last constant (if \( g \equiv 0 \)) is not comparable with the corresponding constant
\[
K^{3K+1/2}(K-1/K)^{1/2}
\]
obtained in the same paper [20] (it is better if \( K \) is large enough but it is not for \( K \) close to 1). It seems that in the proof of Theorem 3.1 there is some small place for improvement of \( M' \) (taking \( \nu_0 = 1 - K^{-1} \)).

3.1. Appendix. Let us prove that \( m_2 \geq m_1 \), where \( m_1 \) and \( m_2 \) are defined in (3.27) and (3.28). Since \( (3 \cdot (3^2 + 3 - 1))^{3/2} > 46 \), the inequality follows directly if \( K \geq 3 \).

Assume now that \( 1 \leq K \leq 3 \). First of all we have
\[
\frac{46^2}{K^2 46^2 K} - \frac{4^{2(K-1)^2/(1+K)} \cdot K^{-1}(K^2 + K - 1)^{3K}}{K^{1+K}} \geq \frac{1}{K^2} \left( 46^2 - 2^{2(K-1)} \cdot 2^{1/(1-K^2)} \right).
\]
Therefore, the inequality
\[
46^{2K-2} \cdot 2^{1/(1-K^2)} \leq K^8
\]
implies \( m_2 \geq m_1 \).

Let \( K \leq 2 \). Then \( \frac{46}{2^{1+K}} < 16 = 2^4 \). By Bernoulli’s inequality \( 2^{K-1} = (1 + 1)^{K-1} \leq 1 + K - 1 = K \) for \( K \leq 2 \). This yields (3.29).

Assume now that \( 2 \leq K \leq 3 \). Then
\[
\frac{46}{2^{1+K}} < e^2.
\]
Thus
\[
\left( \frac{46}{2^{1+K}} \right)^K \leq e^{2(K-1)}.
\]
Therefore, if we prove
\[
K^{-1} \leq K^2 \text{ for } 2 \leq K \leq 3
\]
we will prove the inequality \( m_2 \geq m_1 \) completely.

Let \( x = K - 1 \). Then
\[
K^2 - e^{K-1} = 1 + 2x + x^2 - 1 - x - x^2/2 - x^3/3! - x^4/4! - \ldots
\]
\[
= x(1 + x/2 - x^2/3! - x^3/4! - \ldots)
\]
\[
\geq x(1 - x^3/4! - \ldots)
\]
\[
\geq x(1 - 0.5(e^2 - 1 - 2 - 2^2/2 - 2^3/6)) > x/2,
\]
as desired.

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References