ON QUASICONFORMAL HARMONIC MAPS BETWEEN SURFACES

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ABSTRACT. The following theorem is proved: If $w$ is a quasiconformal harmonic mapping between two Riemann surfaces with compact and smooth boundaries and approximate analytic metrics, then $w$ is bi-Lipschitz continuous with respect to internal metrics. If the surfaces are subsets of the Euclidean spaces, then $w$ is bi-Lipschitz with respect to the Euclidean metrics.

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1. INTRODUCTION

1.1. The main definitions and notation. By $\mathbb{U}$ is denoted the unit disk, and by $S^1$ is denoted its boundary. By $D$ and $\Omega$ are denoted domains in complex plane.

Let $(\Sigma_1, \sigma)$ and $(\Sigma_2, \rho)$ be Riemann surfaces (with or without boundary), with metrics $\sigma$ and $\rho$ respectively. We say that a mapping $w$ between Riemann surfaces $(\Sigma_1, \sigma)$ and $(\Sigma_2, \rho)$ is bi-Lipschitz, if there exist constants $q > 0$ and $Q > 0$, such that

$$qd_\sigma(z_1, z_2) \leq d_\rho(w(z_1), w(z_2)) \leq Qd_\sigma(z_1, z_2), \quad z_1, z_2 \in \Sigma_1,$$

where

$$d_\varrho(z_1, z_2) = \inf_{\gamma \in \gamma_{z_1, z_2}} \int_\gamma \varrho(z) |dz|, \quad \varrho \in \{\sigma, \rho\},$$

and

$$\Gamma_{z_1, z_2} = \{\gamma : \gamma \text{ is a rectifiable curve joining } z_1 \text{ and } z_2 \text{ in } \Sigma_1\}.$$
If \( f : (\Sigma_1, \sigma) \rightarrow (\Sigma_2, \rho) \) is a \( C^2 \) mapping, then \( f \) is said to be harmonic with respect to \( \rho \) (abbreviated \( \rho \)-harmonic) if

\[
(1.1) \quad f_{zz} + (\log \rho^2)_w \circ f f_z \overline{f_z} = 0,
\]

where \( z \) and \( w \) are the local parameters on \( \Sigma_1 \) and \( \Sigma_2 \) respectively. From (1.1) we see that, the harmonicity of \( f \) depends only on the conformal structure, but not on the particular metric of \( \Sigma_1 \).

Also \( f \) satisfies (1.1) if and only if its H. Hopf differential

\[
(1.2) \quad \Psi = \rho^2 \circ f f_z \overline{f_z}
\]

is a holomorphic quadratic differential on \( \Sigma_1 \).

For \( g : \Sigma_1 \rightarrow \Sigma_2 \) the energy integral is defined by

\[
(1.3) \quad E[g, \rho] = \int_{\Sigma_1} \rho^2 \circ g(|\partial g|^2 + |\bar{\partial} g|^2) dV_\sigma,
\]

where \( \partial g \) and \( \bar{\partial} g \) are the partial derivatives taken with respect to the metrics \( \rho \) and \( \sigma \), and \( dV_\sigma \) is the volume element on \( (\Sigma_1, \sigma) \). Assume that energy integral of \( f \) is bounded. Then \( f \) is harmonic if and only if \( f \) is a critical point of the corresponding functional, where the homotopy class of \( f \) is the range of this functional.

We will consider harmonic mappings between compact Riemann surfaces with boundaries, with respect to a metric \( \rho \), where the metric \( \rho \) satisfies the following inequality

\[
|(\log \rho^2)_w| = \frac{|\nabla \rho|}{\rho} \leq M,
\]

where \( M \) is a constant (with respect to local parameters). Under this condition, if for example the domain of \( \rho \) is the unit disk, then there hold the double inequality

\[
(1.4) \quad \rho(0)e^{-M} \leq \rho(w) \leq \rho(0)e^M.
\]

Such metrics are called approximately analytic \([32]\). The spherical metric

\[
\rho(w) = \frac{2}{1 + |w|^2}
\]

is approximately analytic, but the hyperbolic metric

\[
(1.5) \quad \lambda(w) = \frac{2}{1 - |w|^2}
\]

is not. Let us mention the following important fact. (1.1) is equivalent to the following system of equations, which can be directly extended to the dimensions bigger than 2:

\[
(1.6) \quad \Delta u^i + \sum_{\alpha, \beta, k, \ell=1}^2 \Gamma^i_{k\ell}(u) D_\alpha u^k D_\beta u^\ell, \quad i = 1, 2 \quad (f = (u^1, u^2))
\]
where $\Gamma^i_{k\ell}$ are Christoffel Symbols of the metric $\rho$ (or of a metric tensor $(g_{j\ell})$):

$$
\Gamma^i_{k\ell} = \frac{1}{2} g^{im} \left( \frac{\partial g_{m\ell}}{\partial x^k} + \frac{\partial g_{m\ell}}{\partial x^k} - \frac{\partial g_{k\ell}}{\partial x^m} \right) = \frac{1}{2} g^{im} (g_{mk,\ell} + g_{m\ell,k} - g_{k\ell,m}),
$$

and the matrix $(g^{jk})$ is an inverse of the metric tensor $(g_{j\ell})$.

It can be easily seen that, since (1.6) and (1.1) are equivalent, a metric $\rho$ is approximate analytic if and only if Christoffel symbols are bounded.

Let $P$ be the Poisson kernel, i.e. the function

$$
P(z, e^{i\theta}) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2},
$$

and let $G$ be the Green function of the unit disk with respect to the Laplace operator, i.e. the function

$$
G(z, w) = \frac{1}{2\pi} \log \frac{|1 - zw|}{|z - w|}, \quad z, w \in \mathbb{U}, \quad z \neq w.
$$

Let $f : S^1 \to \mathbb{C}$ be a bounded integrable function on the unit circle $S^1$ and let $g : U \to \mathbb{C}$ be continuous. The solution of the equation $\Delta w = g$ (in weak sense) in the unit disk satisfying the boundary condition $w|_{S^1} = f \in L^1(S^1)$ is given by

$$
w(z) = P[f](z) - G[g](z)
$$

(1.8)

$$
:= \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\phi}) f(e^{i\phi}) d\phi - \int_{\mathbb{U}} G(z, \omega) g(\omega) \, dm(\omega),
$$

$|z| < 1$, where $dm(\omega)$ denotes the Lebesgue measure in the plane. It is well known that, if $f$ and $g$ are continuous in $S^1$ and in $\overline{U}$ respectively, then the mapping $w = P[f] - G[g]$ has a continuous extension $\hat{w}$ to the boundary, and $\hat{w} = f$ on $S^1$.

See [34, pp. 118–120].

Let $0 \leq k < 1$ and let $K = \frac{1+k}{1-k}$. An orientation preserving diffeomorphism $w$ between two Riemann surfaces is called $K$–quasiconformal (abbreviated q.c.) if

$$
|w_{\bar{z}}| \leq k|w_z|
$$

(in local coordinates). The previous inequality can be written us

$$
|\nabla w(z)| \leq Kl(\nabla(w(z))),
$$

where

$$
|\nabla w(z)| := \sup \{ |\nabla w(z)h| : |h| = 1 \} = |w_z| + |w_{\bar{z}}|
$$

and

$$
l(\nabla w(z)) := \inf \{ |\nabla w(z)h| : |h| = 1 \} = |w_z| - |w_{\bar{z}}|.
$$

See [1] and [38] for the definition of arbitrary quasiconformal mapping between plane domains, Euclidean surfaces or Riemann surfaces.
1.2. **Background.** In this background, some later results concerning local and global behaviors of harmonic homeomorphisms are given. If $\sigma$ is the Euclidean metric and $w$ is a harmonic mapping defined in a simply connected domain $\Omega$, then $w = g + h$, where $g$ and $h$ are analytic in $\Omega$. If $w$ is an orientation preserving homeomorphism, then by Lewy’s theorem (\cite{19}), $J_w(z) := |g'|^2 - |h'|^2 > 0$. This infer that the analytic mapping $a = \frac{h'}{g'}$ is bounded by 1 in $\Omega$. Moreover, if $w$ is an Euclidean harmonic mapping of the unit disk $\mathbb{U}$ onto convex Jordan domain $\Omega$, mapping the boundary $\partial \mathbb{U}$ onto $\partial \Omega$ homeomorphically, then $w$ is a diffeomorphism. This is an old theorem of Rado, Knesser and Choquet (\cite{17}).

This has been extended in various directions. In \cite{4} it was extended to multiply connected domain as follows. Let $D$ be a bounded finitely connected Jordan domain, and let $\Omega$ be a bounded domain whose inner boundary components are locally connected. Suppose $f^*$ is a sense-preserving weak homeomorphism of $\partial D$ onto $\partial \Omega$, and let $f$ be the harmonic extension of $f^*$ to $D$. If $f(D) \subset \Omega$, then $f$ maps $D$ univalently onto $\Omega$. Conversely, if $f$ is univalent in $D$, then $f(D) = \Omega$. On the other hand, concerning the harmonic maps (with non-Euclidean metric), the result has been extended by Shoen and Yau \cite{39} as follows. Let $\Sigma_1$ be a compact Riemann surface with boundary and with interior $\Sigma_0^1$. Let $\Sigma_2$ be a compact Riemann surface (with or without boundary), and let $\varphi : \Sigma_1 \rightarrow \Sigma_2$ be a diffeomorphism of $\Sigma_1^0$ onto its image, and a homeomorphism of $\partial \Sigma_1^0$ onto its image. Suppose $\Sigma_2$ has a metric $\rho(z) |dz|^2$ of nonpositive curvature, and that $\varphi(\partial \Sigma_1)$ is a union of curves each having nonnegative geodesic curvature with respect to $\varphi(\Sigma_1)$. There exists a unique map $f : \Sigma_1 \rightarrow \Sigma_2$ which is harmonic with respect to $\rho$, which is a diffeomorphism of $\Sigma_1^0$ onto its image, and which satisfies the conditions: $f = \varphi$ on $\partial \Sigma_1$ and $f$ homotopic to $\varphi$ relative to $\partial \Sigma_1$. Later J. Jost (\cite{8}) obtained the following extension: Let $\Sigma_1$ and $\Sigma_2$ be suitably smooth Riemannian manifolds, $\Sigma_1$ simply connected with boundary $\partial \Sigma_1 \in C^{2,\alpha}$ and with interior $\Sigma_1^0$, $\Sigma_2$ without boundary. Let $\psi : \partial \Sigma_1 \rightarrow \Sigma_2$ be a continuous mapping. Consider the Dirichlet problem of finding a harmonic map $\varphi : \Sigma_1 \rightarrow \Sigma_2$ with the given boundary values: $\varphi|\partial \Sigma_1 = \psi$. Suppose that $\psi(\partial \Sigma_1)$ is contained in a geodesic ball $B_M(p)$ with: (i) radius $M < \pi/(2 \sqrt{\kappa})$ where $\kappa > 0$ is an upper bound for the sectional curvatures of $B_M(p)$; (ii) the cut locus of the center $p$ disjoint from $B_M(p)$. If $\Sigma_1$ and $\Sigma_2$ are two-dimensional and $\psi : \partial \Sigma_1 \rightarrow \Sigma_2$ is a homeomorphism onto a sufficiently smooth convex curve, the above solution $\varphi$ is a homeomorphism in $\Sigma_2$ and diffeomorphism in $\Sigma_1^0$.

The Rado-Knesser-Choquet theorem has been extended to the solutions of quasilinear degenerate elliptic equations of the type of the $p$-Laplacian, Alessandrini and Sigalotti (\cite{2}). Schulz (\cite{40}) proved the following extension of Lewy’s theorem. Let $u = (u_1, u_2)$ be a homeomorphism with a finite Dirichlet integral, which solve the Heinz-Lewy quasilinear elliptic system

$$
L(u_k) = \sum_{\alpha, \beta, i, j} c^{\alpha \beta} h_{ij}^k(u) D_{\alpha} u^i D_{\beta} u^j \quad (k = 1, 2)
$$

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in a domain \( \Omega \subset \mathbb{R}^2 \), where

\[
L = -(I/a(u)) \sum_{\alpha=1}^{2} D_\alpha(a(u)D\alpha),
\]

\( c^{\beta \beta} = c \) for \( \beta = 1, 2 \), and \( c^{12} = d, c^{21} = -d \) for some real constants \( c, d \). Under the assumption that \( a(u) \) is Hölder continuous and \( h_{ij}^k \) are Lipschitz continuous, the Jacobian of homeomorphic solutions of (1.9) is nonvanishing. See also [3].

In private conversation I was told that O. Martio gave the first counterexample of extension of Rado’s theorem to higher dimensions ([26]). Let us also quote the recent interesting counterexamples by Melas [27] and by Laugesen [20] to the extension of Rado’s theorem to higher dimensions. Note finally that Lewy’s theorem fails in higher dimensions ([45]).

In this paper we deal with q.c. \( \rho \) harmonic mappings and study their global Lipschitz character. See [25] for the pioneering work on this topic and see [31] for related earlier results. In some recent papers, a lot of work has been done on this class of mappings ([9]-[16], [36], [35]). In these papers is established the bi-Lipschitz character of q.c. harmonic mappings between plane domains with certain boundary conditions. The most important results of these papers is that, Euclidean harmonic q.c. mappings between plane domains with smooth boundaries are Euclidean quasi-isometries (and consequently hyperbolic quasi-isometries). Notice that, in general, quasi-symmetric selfmappings of the unit circle do not provide quasiconformal harmonic extension to the unit disk. In [25] is given an example of \( C^1 \) diffeomorphism of the unit circle onto itself, whose Euclid harmonic extension is not Lipschitz.

In contrast to the case of Euclidean metric, in the case of the hyperbolic metric, if \( f : S^1 \to S^1 \) is a \( C^1 \) diffeomorphism, or more general if \( f : S^{n-1} \to S^{n-1} \) is a mapping with a non-vanishing energy, then its hyperbolic harmonic extension is \( C^{1} \) up to the boundary ([21]) and ([22]). On the other hand Wan ([43]) showed that, q.c. hyperbolic harmonic mappings between smooth domains are hyperbolic quasi-isometries (but in general they are not Euclidean quasi-isometries, neither its boundary values is absolutely continuous, in general). See also [23] for the generalization of the last result to hyperbolic Hadamard surfaces.

1.3. **Main results.** The starting position of this paper are the following recent results.

**Proposition 1.1.** [13] Let \( w \) be a quasiconformal \( C^1 \) homeomorphism from a bounded plane domain \( D \) with \( C^{1,\alpha} \) boundary onto a bounded plane domain \( \Omega \) with \( C^{2,\alpha} \) boundary. If there exist constants \( B \) and \( C \) such that

\[
\Delta w \leq B|\nabla w|^2 + C, \quad z \in D \text{ (in weak sense)},
\]

then \( w \) has bounded partial derivatives in \( D \).

In [12] is proved the weaker version of Proposition 1.1, but which can be also used in our context.
Proposition 1.2. [16] Let \( w = f(z) \) be a \( K \) quasiconformal harmonic mapping between a Jordan domain \( D \) with \( C^{1,\alpha} \) boundary and a Jordan domain \( \Omega \) with \( C^{2,\alpha} \) boundary. Let in addition \( a \in D \) and \( b = f(a) \). Then \( w \) is bi-Lipschitz.

Using a different approach, we extend the last result to the class of harmonic mappings with respect to approximate analytic metrics and harmonic mappings between Riemann surfaces. The following three theorems will be proved in this paper.

Theorem 1.3 (The main theorem). If \( w \) is a \( C^1 \) \( K \)-quasiconformal mapping of the unit disk onto itself, satisfying the inequality

\[
|\Delta w| \leq B|\nabla w|^2
\]

i.e.

\[
\Delta w = -W, \quad |W(z)| \leq B|\nabla w|^2 \quad (z \in \mathbb{U}), \text{ in the weak sense. Then, } w \text{ is bi-Lipschitz.}
\]

Theorem 1.4. Assume that \( \rho \) is an approximate analytic metric and let \( w \) be a \( \rho \) harmonic q.c. selfmapping of the unit disk. Let \( z_n \) be any sequence of points of the unit disk and let in addition \( p_n \) and \( q_n \) be Möbius transformations such that

\[
p_n(w(z_n)) = 0 \quad \text{and} \quad q_n(0) = z_n.
\]

Then there exists a subsequence of \( w_n = p_n \circ w \circ q_n \) converging to a \( \rho_0 \) harmonic mapping \( w_0 \), where \( \rho_0 \) is a metric in the unit disk.

Theorem 1.5. Let \((\Sigma_1, \sigma)\) and \((\Sigma_2, \rho)\) be Riemann surfaces with smooth compact boundaries, with approximate analytic metrics \( \rho \) and \( \sigma \). If \( w : \Sigma_1 \to \Sigma_2 \) is a q.c. harmonic mapping, then \( w \) is bi-Lipschitz.

Together with this introduction the paper contains two other sections. The proof of main theorem (Theorem 1.3) is given in the second section. Let us briefly explain the idea of the proof. Since by Proposition 1.1, a mapping \( w \) satisfying the condition of Theorem 1.3 is Lipschitz, all we need to show is the fact that it is co-Lipschitz, i.e. its inverse mapping is Lipschitz. In order to show that the mapping \( w \) is co-Lipschitz, we will argue by contradiction, which means that there exist a sequence of points \( z_n \) from the unit disk such that \( \lim_{n \to \infty} \nabla w(z_n) = 0 \). In order to do so, previously, we prove a version of Schwartz lemma for harmonic q.c. mappings (Lemma 2.3). Then we take \( w_n = p_n \circ w \circ q_n \), where \( p_n \) and \( q_n \) are Möbius transformations of the unit disk onto itself such that, \( p(0) = 0 \) and \( q(0) = z_n \). \( w_n \) is a sequence of q.c. harmonic mappings with respect to certain metrics \( \rho_n \) satisfying the normalization condition \( w_n(0) = 0 \). This sequence converges, up to some subsequence, to a q.c. harmonic mapping \( w_0 \) with respect to a metric \( \rho_0 \). In proving the last fact we will make use of Arzela-Ascoli theorem, Vitali theorem and of representation formula (1.8). To do so, we will prove more, we will show that, the sequence \( w_n \), together with its gradient \( \nabla w_n \) converges to \( w_0 \) and \( \nabla w_0 \) respectively, uniformly in compact subsets of the unit disk. Several time we will make use of Lemma 2.3 and Proposition 1.1. The fact that \( w_0 \) is q.c. having a critical point and satisfying certain conditions, will contradict Carleman-Hartman-Wintner lemma. The sequence \( w_n \) converges, up to some
subsequences, to some q.c. harmonic mapping $w_0$ independently on the condition
$\lim_{n \to \infty} \nabla w(z_n) = 0$. This procedure, together with the Montel’s theorem for the
Hopf differentials of the sequence $w_n$, will produce a new metric $\rho_0$ and a $\rho_0$-q.c.
harmonic mapping $w_0$. This yields the proof of Theorem 1.4. Together with this
proof, the last section contains the proof of Theorem 1.5.

In the end of the paper it is shown that, this method works for hyperbolic metrics
as well (which are not approximate analytic). Hyperbolic metric and Euclidean
metrics are not bi-Lipschitz equivalent, and therefore q.c. hyperbolic harmonic
mappings are not, in general ordinary bi-Lipschitz mappings.

The conclusion is that, every q.c. harmonic mappings between two Riemann
surfaces is bi-Lipschitz with respect to their corresponding metrics. It remains an
open problem if the quasi-conformality is important in some results we prove.

In the following example it is shown that the inequality (1.11) in the main theo-
rem cannot be replaced by the weaker one (1.10).

**Example 1.6.** [10] Let $w(z) = |z|^\alpha z$, with $\alpha > 1$. Then $w$ is a twice differentiable
$(1 + \alpha)$—quasiconformal self-mapping of the unit disk. Moreover
$$\Delta w = \alpha(2 + \alpha)\frac{|z|^\alpha}{z} = g.$$  
Thus $g = \Delta w$ is continuous and bounded by $\alpha(2 + \alpha)$. However $w^{-1}$ is not
Lipschitz, because $l(\nabla w)(0) = |w_z(0)| - |w_{\bar{z}}(0)| = 0$.

2. **The Proof of Main Theorem**

The following important lemma lies behind our main results.

**Proposition 2.1 (The Carleman-Hartman-Wintner lemma).** [28] Let $\varphi \in C^1(D)$
be a real-valued function satisfying the differential inequality
$$|\Delta \varphi| \leq C(|\nabla \varphi| + |\varphi|)$$
i.e.,
$$\Delta \varphi = -W, \quad |W(z)| \leq C(|\nabla \varphi| + |\varphi|)$$
$(z \in D)$, in the weak sense. Suppose that $D$ contains the origin. Assume that
$\varphi(z) = o(|z|^n)$ as $|z| \to 0$ for some $n \in N_0$. Then
$$\lim_{z \to 0} \frac{\varphi_z(z)}{z^n}$$
exists.

The following proposition is a consequence of Carleman-Hartman-Wintner lemma.

**Proposition 2.2.** [41, Proposition 7.4.3.] Let $\{u_k(z)\}$ be a sequence of real func-
tions of class $C^1(D)$ satisfying the differential inequality

(2.1) $$|\Delta u_k| \leq C(|\nabla u_k| + |u_k|)$$

where $C$ is independent of $k$. Assume that

(2.2) $$u_k(z) \to u_0(z), \quad \nabla u_k(z) \to \nabla u_0(z),$$
uniformly in $D$ ($k \to \infty$). Assume in addition

(2.3) \quad u_0(z) = o(|z|) \text{ as } |z| \to 0,$

and that

(2.4) \quad \nabla u_k(z) \neq 0 \text{ for all } k \text{ and } z \in D.$

Then $u_0(z) \equiv 0.$

The proof of the main theorem is based on the following three lemmas.

**Lemma 2.3.** If $w : \mathbb{U} \to \mathbb{U}, w(0) = 0,$ satisfies the conditions of Theorem 1.3, then there exists a constant $C(K)$ such that

(2.5) \quad \frac{1 - |z|^2}{1 - |w(z)|^2} \leq C(K) \quad z \in \mathbb{U}.

**Proof.** Take

$$QC(\mathbb{U}, B, K) = \{ w : \mathbb{U} \to \mathbb{U} : w(0) = 0, |\Delta w| \leq B|\nabla w|^2, w \text{ is K.q.c.} \}.$$ Let us choose $A$ such that the function $\varphi_w, w \in QC(\mathbb{U}, B, K)$ defined by

$$\varphi_w(z) = -\frac{1}{A} + \frac{1}{A}e^{A(|w(z)|-1)}$$
is subharmonic in $\varrho := 4^{-K} \leq |z| \leq 1.$

Take

$$s = \frac{w}{|w|}, \quad \rho = |w|.$$

As $w = s\rho$ is a $K$ quasiconformal selfmapping of the unit disk with $w(0) = 0,$ by Mori’s theorem ([44]) it satisfies the doubly inequality:

(2.6) \quad \frac{z}{4^{1-1/K}} \leq \rho \leq 4^{1-1/K}|z|^{1/K}.$

By (2.6) for $\varrho \leq |z| \leq 1$ where

(2.7) \quad \varrho := 4^{-K}
we have

(2.8) \quad \rho \geq \rho_0 := 4^{1-K^2-K}.$

Now we choose $A$ such that

$$\frac{A\rho_0^2}{K^2} + 2 - 2BK^2 \geq 0.$$ Take

$$\chi(\rho) = -\frac{1}{A} + \frac{1}{A}e^{A(\rho-1)}.$$ Then

$$\chi'(\rho) = e^{A(\rho-1)}$$ and

$$\chi''(\rho) = Ae^{A(\rho-1)}.$$
On the other hand

\( (2.9) \quad \Delta \varphi_w(z) = \chi''(\rho)|\nabla|w|^2 + \chi'(\rho)\Delta |w|. \)

Furthermore

\( (2.10) \quad \Delta |w| = 2|\nabla s|^2 + 2 \langle \Delta w, s \rangle. \)

To continue observe that

\( (2.11) \quad \nabla w = \rho \nabla s + \nabla \rho \otimes s. \)

Since

\[ |\nabla w(z)| \leq Kl(\nabla w(z)), \]

choosing appropriate unit vector \( h \) we obtain the inequality

\( (2.12) \quad |\nabla w| \leq K(\rho)|\nabla s|. \)

Similarly it can be proved the inequality

\( (2.13) \quad K|\nabla w| \geq \rho|\nabla s|. \)

Using (2.8), (2.9), (2.10), (1.11), (2.12) and (2.13), it follows finally that

\[ \Delta \varphi_w(z) \geq \left( \frac{A\rho^2}{K^2} + 2 - 2BK^2 \right)e^A(\rho^{-1}) - 0, \quad 4^{-K} \leq |z| \leq 1. \]

Define

\[ \gamma(z) = \sup\{ \varphi_w(z) : w \in QC(U, B, K) \}. \]

Prove that \( \gamma \) is subharmonic for \( 4^{-K} \leq |z| \leq 1. \) In order to do so, we will first prove that \( \gamma \) is continuous. For \( z, z' \in U \) and \( w \in QC(U, B, K) \), according to Mori’s theorem (see e.g. [1]), we have

\[ |\varphi_w(z) - \varphi_w(z')| = \frac{1}{A} |(e^A(|w(z)|^{-1}) - e^A(|w(z')|^{-1}))| \]

\[ \leq |w(z) - w(z')| \leq 16 |z - z'|^{1/K}. \]

Therefore

\[ |\gamma(z) - \gamma(z')| \leq 16 |z - z'|^{1/K}. \]

This means in particular that \( \gamma \) is continuous. It follows that \( \gamma \) is subharmonic as the supremum of subharmonic functions (see e.g. [33, Theorem 1.6.2]).

If \( |z_1| = |z_2| \) then \( \gamma(z_1) = \gamma(z_2) \). In order to prove the last statement we do as follows. For every \( \varepsilon > 0 \) there exists some \( w \in QC(U, B, K) \) such that

\[ \varphi_w(z_2) \leq \gamma(z_2) \leq \varphi_w(z_2) + \varepsilon. \]

Now \( w_1(z) = w(\frac{z}{|z|^2}) \) is in the class \( QC(U, B, K) \). Therefore

\[ \varphi_{w_1}(z_1) \leq \gamma(z_1) \leq \varphi_{w_1}(z_1) + \varepsilon. \]

As \( \varepsilon \) is arbitrary and as \( w_1(z_1) = w(z_2) \) it follows that \( \gamma(z_1) = \gamma(z_2) \).

This yields that

\[ \gamma(z) = g(r) = -\frac{1}{A} + \frac{1}{A} e^{A(h(r)-1)}. \]
It is well known that, a radial subharmonic function is an increasing convex function of \( t = \log r \), for \(-\infty < t < 0\) (see [37, Theorem 2.6.6]). From (2.6)

\[
g(4-K) \leq -\frac{1}{A} + \frac{1}{A}e^{A(4^{-1/K}-1)} < 0 = g(1),
\]

it follows that \( \gamma \) is nonconstant. Since \( \gamma \) is a subharmonic increasing convex function of \( \log r \), it follows that, for \( r > s \)

\[
g'(r+0) \geq g'(r-0) \geq g'(s+0) \geq g'(s-0) \geq 0.
\]

Since it is non-constant, it satisfies in particular that,

\[
g'(1-0) > 0.
\]

Notice that, the last inequality is also a consequence of E. Hopf boundary point lemma, see e.g. [6].

Therefore

\[
-\frac{1}{A} + \frac{1}{A}e^{A(|w(z)|-1)} \leq -\frac{1}{A} + \frac{1}{A}e^{A(h(r)-1)},
\]

i.e.

\[
|w(z)| \leq h(r), |z| = r, w \in QC(U, B, K),
\]

where

\[
h(r) < 1 \text{ and } h'(1-0) > 0.
\]

It follows that

\[
\frac{1 - |z|^2}{1 - |w(z)|^2} \leq \frac{1 - |z|^2}{1 - |h(|z|)|^2} \leq C(K).
\]

\[\square\]

**Remark 2.4.** The previous lemma can be considered as a Schwartz lemma for the class \( QC(U, B, K) \). Let

\[
F(U, B) = \{ w : U \to \mathbb{U} : w(0) = 0, |\Delta w| \leq B|\nabla w|^2 \}.
\]

Then for \( B = 0 \) the class \( F(U, B) \) coincides with the class of harmonic functions of the unit disk into itself satisfying the normalization \( w(0) = 0 \). Using Schwartz lemma for harmonic functions, it can be shown that, there exists a constant \( C \) such that

\[
(2.14) \quad \frac{1 - |z|^2}{1 - |w(z)|^2} \leq C, \quad w \in F(U, 0).
\]

It would be of interest to verify if quasiconformality is important for \( QC(U, B, K) \).

In other word do there hold (2.14) for the class \( F(U, B) \).

**Lemma 2.5.** Let \((z_n)\) be an arbitrary sequence of complex numbers from the unit disk. Assume that \( w \) satisfies the conditions of Theorem 1.3. Let \( p_n \) and \( q_n \) be Möbius transformations, of the unit disk onto itself such that, \( p(w(z_n)) = 0 \) and \( q(0) = z_n \). Take \( w_n = p_n \circ w \circ q_n \).

Then, up to some subsequence, which is also denoted by \((w_n)\) we have
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\( |\nabla w_n| \leq \frac{C_1(K)}{1 - |z|}, \)

\( |\Delta w_n| \leq \frac{C_2(K)}{(1 - |z|)^2}, \)

and

\( |\Delta w_n| \leq \frac{C_3(K)}{(1 - |z|)^2} |w_{n\bar{z}}| \cdot |w_{nz}|. \)

**Proof.** Take

\( \frac{p_n}{1 - w w(z_n)} \)

and

\( \frac{q_n}{1 + z \bar{z}_n} \)

It is evident that

\( w_n(z) = p_n \circ w \circ q_n \)

is a \( K \)-q.c. mapping of the unit disk onto itself. By [5] for example, a subsequence of \( w_n \), also denoted by \( w_n \), converges uniformly to a \( K \)-quasiconformal mapping \( w_0 \) of the closed unit disk onto itself.

Next we have

\( (w_n)_z = p'_n w_{q_n} q'_n \)

and

\( (w_n)_{\bar{z}} = p'_n w_{\bar{q}_n} \bar{q}'_n. \)

Using now

\( w_{zz} + 2\partial \log \rho \circ w w_{z\bar{z}} = 0, \)

we derive

\( (w_n)_{z\bar{z}} = ((p_n \circ w \circ q_n)_z)_{\bar{z}} \)

\( = (p'_n w_{q_n} q'_n)_z = p''_n w_{q_n} q'_n + p'_n w_{q_n} q_n \bar{q}'_n \)

\( = ( \frac{p''_n}{p'_n z} - 2 \partial \log \rho ) w_{q_n} w_{\bar{q}_n}. \)

Therefore

\( |(w_n)_{z\bar{z}}| \leq |q'_n|^2 \left( |p''_n| + 2|p'_n| |\partial \log \rho| \right) |w_{q_n}| |w_{\bar{q}_n}|. \)
and

\[ w_{nz \bar{z}} + \left( -\frac{p'''_n}{p''_n} + \frac{2 \partial \log \rho}{p'_n} \right) w_{n \bar{z} \bar{z}} = 0. \]  

Now we have

\[ |q'_n| = \frac{1 - |z_n|^2}{1 + z_n \bar{z}_n} = \frac{1 - |q_n(z)|^2}{1 - |z|^2}, \]

\[ |p'_n| = \frac{1 - |w(z_n)|^2}{|1 - w(q_n(z))w(z_n)|^2} = \frac{1 - |p_n(w(q_n(z)))|^2}{1 - |w(q_n(z))|^2} \]

and

\[ |p''_n| = \frac{(1 - |w(z_n)|^2)|w(z_n)|}{|1 - w(q_n(z))w(z_n)|^3}. \]

From (2.17) – (2.23) and (2.5) we obtain

\[ |(w)_{z \bar{z}}| \leq \frac{C(K)}{1 - |z|}, \quad |(w)_{\bar{z} z}| \leq \frac{C(K)}{1 - |z|} \]

and

\[ |q'_n|^2 \left( |p''_n| + 2|p'_n||\partial \log \rho| \right) \leq 2 \left( 1 - |z_n|^2 \right)^2 \left( 1 + |\partial \log \rho| \right) \left( \frac{1}{(1 - |z_n|^2)^2} \right). \]

Combining (2.5), (2.19) and (2.25) we obtain

\[ |(w)_{z \bar{z}}| \leq \frac{2C(K)^2(1 + |\partial \log \rho|)}{(1 - |z|)^4}. \]

Let us estimate the sequence

\[ S_n = -\frac{p'''_n}{p''_n} + \frac{2 \partial \log \rho}{p'_n}. \]

First of all

\[ \frac{p'''_n}{p''_n} = \frac{2w(z_n)(1 - w_n(z)w(z_n))}{1 - |w(z_n)|^2}. \]

Hence

\[ \left| \frac{p''_n}{p'_n} \right|^2 = \frac{2|w(z_n)||1 - w_n(z)w(z_n)|}{1 - |w(z_n)|^2} \leq \frac{2|w(z_n)||w(\frac{x + z_n}{1 + z_n}) - w(z_n)|w(z_n)|}{1 - |w(z_n)|^2} + 2. \]

To continue observe that
(2.27) \[ |w(z + z_n) - w(z_n)| \leq |\nabla w|_\infty |z| |1 - |z_n|^2| \leq |\nabla w|_\infty |z| |1 - |z_n|^2| |1 + z_n|^{-2}. \]

Thus, by using (2.5) we get

\[ |\frac{p''_n}{p_n^2}| \leq 2 + \frac{|z| |\nabla w|_\infty 1 - |z_n|^2}{|1 - |z}| \frac{2}{1 - |z_n|^2} \leq 2 + C(K) |\nabla w|_\infty \frac{2}{(1 - |z|)^2}, \]

i.e.

(2.28) \[ |\frac{p''_n(w(q_n(z)))}{p_n(w(q_n(z)))^2}| \leq 2 + C(K) |\nabla w|_\infty \frac{2}{(1 - |z|)^2}. \]

Similarly, as

\[ \frac{1}{p'_n(w(q_n(z)))} = \frac{(1 - w(q_n(z))w(z_n))^2}{1 - |w(z_n)|^2} \]

we get, according to (2.27) and (2.5), that

(2.29) \[ |\frac{2}{p'_n(w(q_n(z)))}| \leq 2 + C(K) |\nabla w|_\infty \frac{4}{(1 - |z|)^2}. \]

It follows that

\[ |S_n(z)| \leq 4 + C(K) (1 + 2|\partial \log \rho|) |\nabla w|_\infty \frac{2}{(1 - |z|)^2}. \]

Hence the sequence \( w_n \) satisfies the differential inequality

(2.30) \[ |\Delta w_n| \leq \left( 4 + C(K) \left( 1 + 2|\partial \log \rho| \cdot |\nabla w|_\infty \frac{2}{(1 - |z|)^2} \right) \right) |\partial w_n \partial w_n|. \]

We will finish the proof of main result by using the following lemma.

**Lemma 2.6.** Under the conditions of the previous lemma, there exists a subsequence of \( w_n \) converging to a mapping \( w_0 \) in the \( C^1 \) norm uniformly on compact sets of the unit disk.

**Proof of Lemma 2.6.** Let \( 0 < r < 1 \) and take \( \tilde{w}_n(z) = w_n(rz) \), \( z \in \mathbb{U} \).

From (2.26) it follows that \( g_n = \Delta \tilde{w}_n \) is bounded. By (2.24) \( \tilde{w}_n \) is uniformly bounded. According to (1.8) it follows that

\[ \tilde{w}_n(z) = H_n(z) + G_n(z) = P[f_n](z) - G[g_n](z) \]

(2.31) \[ := \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\varphi}) w_n(r e^{i\varphi}) d\varphi - \int_{\mathbb{U}} G(z, \omega) g_n(\omega) \ dm(\omega), \]

\( |z| < 1 \). Here \( H_n \) is a harmonic function taking the same boundary as \( \tilde{w}_n \) in \( S^1 \).

We will prove that, up to some subsequence, the sequence \( \nabla \tilde{w}_n \) converges to \( \nabla \tilde{w}_0 \) in \( \mathbb{R} \mathbb{U} \).
As $\nabla \hat{w}_n$ is uniformly bounded (see Lemma 2.5 a), $|\nabla G_n| \leq \frac{2}{3}|g_n|$ (this inequality has been shown in [10]) and $|g_n| \leq M$ (see Lemma 2.5 c), it follows that the family of harmonic maps $\nabla H_n$ is uniformly bounded on $U$. Therefore by Cauchy inequality we obtain

\begin{equation}
|\nabla^2 H_n| \leq C \frac{|\nabla H_n|_\infty}{1 - |z|} \leq \frac{C_1}{1 - |z|}.
\end{equation}

To continue observe that, for $z \neq \omega$ we have

\begin{align*}
G_z(z, \omega) &= \frac{1}{4\pi} \left( \frac{1}{\omega - z} - \frac{\bar{\omega}}{1 - z\bar{\omega}} \right) \\
&= \frac{1}{4\pi} \left( 1 - |\omega|^2 \right) (z - \omega)(\bar{z}\bar{\omega} - 1),
\end{align*}

and

\begin{align*}
G_{\bar{z}}(z, \omega) &= \frac{1}{4\pi} \left( 1 - |\omega|^2 \right) (\bar{z} - \bar{\omega})(z\omega - 1).
\end{align*}

Prove that the family of functions

\begin{equation}
F_n(z, z') = \partial G[g_n](z) - \partial G[g_n](z')
\end{equation}

is uniformly continuous on $U \times U$.

First of all $|g_n|_U \leq M$.

Then

\begin{equation}
|\partial G[g_n](z) - \partial G[g_n](z')| \\
\leq \Phi(z, z') := M \frac{1}{4\pi} \int_U \left| \frac{1 - |\omega|^2}{(z - \omega)(\bar{z}\bar{\omega} - 1)} - \frac{1 - |\omega|^2}{(z' - \omega)(\bar{z}'\bar{\omega} - 1)} \right| dm(\omega).
\end{equation}

We will prove that $\Phi(z, z')$ is continuous on $U \times U$, and use the fact that $\Phi(z, z) \equiv 0$.

In other world we will prove that

\begin{equation}
\lim_{n \to \infty} (z_n, z'_n) = (z, z') \Rightarrow \lim_{n \to \infty} \Phi(z_n, z'_n) = \Phi(z, z').
\end{equation}

In order to do so, we use the Vitali theorem (see [29, Theorem 26.C]):

\textit{Let $X$ be a measure space with finite measure $\mu$, and let $h_n : X \to \mathbb{C}$ be a sequence of functions that is uniformly integrable, i.e. such that for every $\varepsilon > 0$ there exists $\delta > 0$, independent of $n$, satisfying}

\begin{equation}
\mu(E) < \delta \implies \int_E |h_n| \, d\mu < \varepsilon.
\end{equation}

\textit{Now: if $\lim_{n \to \infty} h_n(x) = h(x)$ a.e., then}

\begin{equation}
\lim_{n \to \infty} \int_X h_n \, d\mu = \int_X h \, d\mu.
\end{equation}
In particular, if 
\[ \sup_n \int_X |h_n|^p \, d\mu < \infty, \quad \text{for some } p > 1, \]
then (†) and (‡) hold.

We will use the Vitali theorem for 
\[ h_n(\omega) = \left| \frac{1 - |\omega|^2}{(z_n - \omega)(\bar{z}_n \bar{\omega} - 1)} - \frac{1 - |\omega|^2}{(z_n' - \omega)(\bar{z}_n' \bar{\omega} - 1)} \right|, \]
defined in the unit disk.

To prove (2.34), it suffices to prove that 
\[ M := \sup_{z,z' \in U} \int_U \left( \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} + \frac{1 - |\omega|^2}{|z' - \omega| \cdot |1 - \bar{z}'\omega|} \right)^p \, dm(\omega) < \infty, \]
for \( p = \frac{3}{2} \).

Let 
\[ I_p(z) := \int_U \left( \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} \right)^p \, dm(\omega). \]

For a fixed \( z \), we introduce the change of variables 
\[ \frac{z - \omega}{1 - \bar{z}\omega} = \xi, \]
or, what is the same 
\[ \omega = \frac{z - \xi}{1 - \bar{z}\xi}. \]

Therefore 
\[ I_p(z) = \int_U \left( \frac{1 - |\omega|^2}{|z - \omega| \cdot |1 - \bar{z}\omega|} \right)^p \, dm(\omega), \]
\[ = \int_U \frac{(1 - |z|^2)^{2-p}(1 - |\omega|^2)^p}{|\xi|^p |1 - \bar{z}\xi|^4} \, dm(\xi), \]
\[ \leq (1 - |z|^2)^{1/2} \int_0^1 \rho^{-1/2}(1 - \rho^2)^{3/2} \int_0^{2\pi} |1 - \bar{z}\rho e^{i\varphi}|^{-4} \, d\varphi \]
\[ \leq (1 - |z|^2)^{1/2} \int_0^1 \rho^{-1/2}(1 - \rho^2)^{3/2} (1 - |z|\rho)^{-3} \, d\rho. \]

From the elementary inequality 
\[ \int_0^1 \rho^{-1/2}(1 - \rho^2)^{3/2} (1 - |z|\rho)^{-3} \, d\rho \leq C(1 - |z|^2)^{-1/2}, \]
it follows that 
\[ \sup_{z \in U} I_p(z) < \infty. \]

Finally, Holder inequality implies 
\[ M \leq 2^{p-1} \sup_{z,z' \in U} (I_p(z) + I_p(z')) < \infty. \]
This means that, \( \Phi \) is uniformly continuous on \( U \times U \). Using the fact that \( \Phi(z, z) \equiv 0 \), it follows that, for \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
|z - z'| \leq \delta \Rightarrow |\partial G[g_n](z) - \partial G[g_n](z')| \leq \Phi(z, z') \leq \varepsilon.
\]

Similarly we obtain that the family

(2.35) \( T_n(z, z') = \bar{\partial} G[g_n] - \bar{\partial} G[g_n] \) is uniformly continuous on \( U \times U \).

By (2.32), (2.33) and (2.35) and Arzela–Ascoli theorem, there exists a subsequence of \( w_n \) which will be also denoted by \( w_n \) converging to \( w_0 \) in \( C^1 \) metric uniformly on the disk \( r^2 U = \{ z : |z| \leq r^2 \} \):

\[
\lim_{n \to \infty} w_n(z) = w_0(z) \quad \text{and} \quad \lim_{n \to \infty} \nabla w_n(z) = \nabla w_0(z) \quad z \in r^2 U.
\]

Using the diagonalisation procedure it follows the desired conclusion.

\[ \square \]

**Proof of Theorem 1.3.** If \( w(a) = 0 \) such that \( a \neq 0 \), then take the mapping \( w(z + z') \) and also denote by \( w \). It is obvious that is satisfies the conditions of our theorem.

Assume that, there exists a sequence of points \( z_n \) such that \( \lim_{n \to \infty} \nabla w(z_n) = 0 \). From Lemma 2.5, if \( |z| \leq r^2 < 1 \), it follows that, there exists a constant \( C_K^1(r) \) such that

(2.36) \( |\Delta w_n| \leq C_K^1(r) |\nabla w_n| \).

Let \( u_n + iv_n = w_n \). Let \( A \) and \( B \) be defined by

\[
A := |\nabla u_n|^2 = 2(\left|u_{nz}\right|^2 + \left|v_{nz}\right|^2) = \frac{1}{2} (|w_{z} + \bar{w}_{nz}|^2 + |w_{n\bar{z}} + \bar{w}_{n\bar{z}}|^2)
\]

and

\[
B := |\nabla v_n|^2 = 2(\left|v_{nz}\right|^2 + \left|v_{nz}\right|^2) = \frac{1}{2} (|w_{n\bar{z}} - \bar{w}_{nz}|^2 + |w_{nz} - \bar{w}_{nz}|^2).
\]

Then

\[
\frac{A}{B} = \frac{|1 + \mu|^2}{|1 - \mu|^2}
\]

where

\[
\mu = \frac{w_{n\bar{z}}}{w_{nz}}.
\]

Since \( |\mu| \leq k \) it follows that

(2.37) \( \frac{(1 - k)^2}{(1 + k)^2} \leq \frac{A}{B} \leq \frac{(1 + k)^2}{(1 - k)^2} \).

From (2.36) and (2.37) it follows that, there exists a constant \( C_K^2(r) \) such that

(2.38) \( |\Delta u_n| \leq C_K^2(r) |\nabla u_n| \).

From Lemma 2.6

\[
\lim_{n \to \infty} ||\nabla w_n - \nabla w_0||_{r^2 U} + ||w_n - w_0||_{r^2 U} = 0.
\]
We next have
\[ \nabla w_n(0) = \frac{1 - |z_n|^2}{1 - |w(z_n)|^2} \left| \nabla w(z_n) \right|. \]

According to (2.5)
\[ \frac{1 - |z_n|^2}{1 - |w(z_n)|^2} \leq C(K). \]

It follows that \( \nabla w_0(0) = 0 \), and consequently
\[ \nabla u_0(0) = 0. \]

Since \( w_n \) is a quasiconformal diffeomorphism, it follows that \( \nabla w_n \neq 0 \). From (2.37) we obtain that \( \nabla u_n \neq 0 \). Thus all the conditions of Proposition 2.2 are satisfied with
\[ D = \{ z : |z| \leq r \} \]
and \( u_n = \Re w_n \). This infers that \( u_0 \equiv 0 \) which is a contradiction, because \( w_0 \) is a quasi-conformal mapping.

The rest of the proof follows from the fact that \( |\nabla w| \) is bounded below and above by positive constants.

□

3. Applications

The mapping \( w_0 \) produced in Lemma 2.6 exists without the a priori assumption that \( \nabla w(z_n) \to 0 \). In the following proof we prove that \( w_0 \) is a harmonic mapping with respect to an appropriate conformal metric \( \rho_0 \) depending on the initial metric \( \rho \) and on the sequence \( z_n \).

Proof of Theorem 1.4. First of all, using (2.17) and (2.18) we have
\[ w_n z \overline{w_n z} = |p'_n|^2 w_{q_n} \overline{w_{q_n} q'_n(z)}^2. \]

On the other hand, since \( w \) is \( \rho \) harmonic it follows that
\[ \Psi_w(q_n(z)) = \rho^2(w(q_n(z))) w_{q_n} \overline{w_{q_n} q'_n(z)}^2 \]
is analytic. Thus \( w_n \) is \( \rho_n \) harmonic for
\[ \rho_n^2(w_n(z)) = \frac{\rho^2(w(q_n(z)))}{|p'_n(w(q_n(z)))|^{2(1 - |z_n|^2)^2}}. \]

This means that the Hopf differential
\[ \Psi_n(z) = \rho_n^2(w_n(z)) w_n z \overline{w_n z} \]
of \( w_n \) is analytic. According to Proposition 1.1 and (2.21) it follows that
\[ |\Psi_n(z)| \leq \frac{C}{(1 - |z|)^4}. \]

Therefore by Montel’s theorem, up to some subsequence it converges to some analytic function \( \Psi_0 \) on the unit disk.

On the other hand, up to some subsequence (according to Lemma 2.6)
\[ w_n z \overline{w_n z} \]
converges uniformly in compact sets of the unit disk to

\[ w_0 \, \Psi \, w_{0z}. \]

Also we have

\[
\rho_n(w_n(z)) = \frac{\rho(w(q_n(z)))}{|p_n'(w(q_n(z)))|} \left(1 - |z_n|^2\right)
\]

\[
\leq C \frac{|1 - w(q_n(z))|}{(1 - |z_n|^2)(1 - |z_n|^2)}
\]

\[
\leq C \left[1 - |w(z_n)|^2 + 2|w(q_n(z)) - w(z_n)|(1 - |z_n|^2) + |w(q_n(z)) - w(z_n)|^2 \right]
\]

\[
\leq C \left[1 - |w(z_n)|^2 + 2|w(q_n(z)) - w(z_n)| + |w(q_n(z)) - w(z_n)|^2 \right].
\]

To continue use again the fact that \(|\nabla w|_\infty < \infty\). Therefore

\[
|w(q_n(z)) - w(z_n)| \leq |\nabla w|_\infty |q_n'(z)||z - 0| = |\nabla w|_\infty \frac{|z|^2(1 - |z_n|^2)}{1 + \pi z_n^2}
\]

and hence

\[
\frac{|w(q_n(z)) - w(z_n)|}{1 - |z_n|^2} \leq |\nabla w|_\infty \frac{1}{(1 - |z|)^4}.
\]

Combining the previous inequalities and (2.5) we obtain

\[
\rho_n(w_n(z)) \leq C \left(2|\nabla w|_\infty + \frac{2|\nabla w|_\infty}{(1 - |z|)^2} + \frac{C(K)|\nabla w|_\infty^2}{(1 - |z|)^2} \right).
\]

It follows that

\[
\rho_n^2(w_n(z)) \to B(z) := \frac{\Psi_0(z)}{w_{0z} \, \Psi_{0z}},
\]

where the quantity

\[
B(z) = \frac{\Psi_0(z)}{w_{0z} \, \Psi_{0z}}
\]

is finite for \(z \in U\).

Thus

(3.2) \quad \rho_n^2(t) \to \rho_0^2(t) := B(w^{-1}_0(t)).

Without loss of generality assume that \(z_n \to 1\). \(\rho_0\) is not identical to zero because, according to (2.5) and (1.4)

\[
\rho_0(0) = \lim_{n \to \infty} \frac{\rho(w(q_n(0)))}{|p_n'(w(q_n(0)))|} \left(1 - |z_n|^2\right)
\]

\[
= \frac{\rho(w(1))}{\lim_{n \to \infty} \frac{1 - |z_n|^2}{1 - |w(z_n)|^2}} \geq \frac{\rho(w(1))}{C(K)} > 0.
\]

This means that \(\rho_0\) is a metric on the unit disk.

We obtain that, \(w_0\) is a harmonic quasiconformal mapping of the unit disk with respect to the metric \(\rho_0\) defined in (3.2). \(\Box\)
The next theorem implies Theorem 1.5.

**Theorem 3.1.** Let $(\Sigma_1, \sigma)$ and $(\Sigma_2, \rho)$ be $C^{2,\alpha}$ surfaces, with $C^{2,\alpha}$ compact boundaries and of equal connectivities, such that $\sigma$ and $\rho$ are approximate analytic metrics. Let $w : \Sigma_1 \to \Sigma_2$ be a harmonic homeomorphism. Then the following conditions are equivalent:

a) $w$ is quasiconformal;

b) $w$ is bi-Lipschitz;

c) $w$ is bi-Lipschitz with respect to Euclidean metrics (with respect to local parameters).

**Proof.** The proof depends on the following Kellogg’s type proposition.

**Proposition 3.2.** [7, Theorem 3.1] Suppose $S$ is a surface with boundary, homeomorphic to a plane domain $G$ bounded by $k$ circles via a chart $\psi : G \to S$. Suppose the coefficients of the metric tensor of $S$ can be defined in this chart by bounded measurable functions $g_{ij}$ with $g_{11} g_{22} - g_{12}^2 \geq \lambda > 0$ in $G$. Then $S$ admits a conformal representation $\tau \in H^2 \cap C^{\alpha}(\bar{B}, \bar{G})$, where $B$ is a plane domain bounded by $k$ circles and $\tau$ satisfies almost everywhere the conformality relations

$$|\tau_x|^2 = |\tau_y|^2, \quad \langle \tau_x, \tau_y \rangle = 0$$

(Here $(x, y)$ denote the coordinates of points in $B$, and norms and products are taken with respect to the metric of $S$).

Furthermore, concerning higher regularity, $\tau$ is as regular as $S$, i.e. if $S$ is of class $C^{m,\alpha}(\bar{B})$ ($m \in \mathbb{N}, 0 < \alpha < 1$) or in $C^\infty$ then also $\tau \in C^{m,\alpha}(\bar{B})$ or $\tau \in C^\infty(\bar{B})$, respectively. In particular, if $S$ is at least $C^{1,\alpha}$ then the conformality relations are satisfied everywhere, and $\tau$ is a diffeomorphism.

We consider four cases.

(i) $\Sigma_1$ and $\Sigma_2$ are compact surfaces without boundary. The theorem is well-known, since every harmonic homeomorphism is a diffeomorphism and consequently it is bi-Lipschitz.

(ii) $\Sigma_1$ and $\Sigma_2$ are conformally equivalent to the unit disk. Then for $i = 1, 2$ there exists a conformal mapping $\tau_i : U \to \Sigma_i$. Let $w$ be $K$-quasiconformal. Take $\hat{w} = \tau_2^{-1} \circ w \circ \tau_1$. Let us show that, $\hat{w}$ is a harmonic mapping of the unit disk onto itself with approximate analytic metric. First of all

$$\tau'_2 \hat{w}_z = \partial w \tau'_1,$$

$$\tau'_2 \hat{w}_{\bar{z}} = \bar{\partial} w \tau'_1,$$

and

$$\tau''_2 \hat{w}_z \cdot \hat{w}_{\bar{z}} + \tau'_2 \hat{w}_{z\bar{z}} = \partial \bar{\partial} |\tau'_1|^2.$$

Thus

$$\frac{\hat{w}_{z\bar{z}}}{w_z \cdot w_{\bar{z}}} = -\frac{\tau''_2}{(h'_2)^2} + \tau'_2 \frac{\partial \bar{\partial} w}{\partial w} \cdot \frac{\partial w}{\bar{\partial} w} = -\frac{\tau''_2}{(h'_2)^2} - 2\tau'_2 \frac{\partial \sigma_2}{\sigma_2}.$$

By using (1.4), it follows that, the coefficients of the metric tensor
\[ g_{ij} = \begin{cases} \rho(z), & \text{if } i = j; \\ 0, & \text{if } i \neq j, \end{cases} \]

satisfy the condition of Proposition 3.2. Therefore \(|\tau''|, |\tau'| \) and \(\frac{1}{|\tau'|^2} \) are bounded. Thus \(\hat{w} \) is a quasiconformal harmonic mapping with respect to an approximate analytic metric. Theorem 1.3 implies that \(\hat{w} \) is bi-Lipschitz with respect to Euclidean metric. Since \(\tau_1 \) and \(\tau_2 \) are diffeomorphisms up to the boundaries, the mapping is bi-Lipschitz as well. By (1.4), for \(\rho \in \{\rho, \sigma\} \) there exists a constant \(P_\rho > 0 \) such that

\[ P_\rho^{-1}|w_1 - w_2| \leq d_\rho(w_1, w_2) \leq P_\rho|w_1 - w_2|, \quad w_1, w_2 \in \Sigma_\rho. \]

Thus internal distance \(d_\rho\), which is induced by the metric \(\rho\) in \(\Sigma_\rho\) and Euclidean metric are bi-Lipschitz equivalent. It follows that \(w\) is bi-Lipschitz (with respect to internal metrics). Therefore \(a) \Rightarrow b) \) and \(b) \Leftrightarrow c) \).

Since every Euclidean bi-Lipschitz is quasiconformal, according to the previous facts we obtain \(b) \Rightarrow a) \).

(iii) \(\Sigma_1 \) and \(\Sigma_2 \) are homeomorphic to a plane domain \(G\) bounded by \(k\) circles. Let \(\tau_1: B \to \Sigma_1\) and \(\tau_2: B \to \Sigma_2\) be conformal mappings produced in Proposition 3.2, where \(B\) is a plane domain bounded by \(k\) circles. Take \(\hat{w} = \tau_2^{-1} \circ w \circ \tau_1\).

For every boundary point \(t \in \partial \Sigma_1\), there exists a neighborhood \(B(t) \subset D\) of \(s = \tau_1^{-1}(t)\) (with respect to the boundary of \(D\)), which is conformally equivalent to the unit disk. Let \(\tau_3: \mathbb{U} \to B(t)\) and \(\tau_4: \mathbb{U} \to \hat{w}(B(t))\) be Riemann conformal mappings. Take now \(\hat{w}_t = \tau_4^{-1} \circ \hat{w} \circ \tau_3\). According to the case (ii), there exists a positive constant \(C_t(K)\) such that:

\[ \frac{1}{C_t(K)} \leq |\nabla \hat{w}_t(z)| \leq C_t(K), \quad z \in \mathbb{U}. \]

Using the Schwarz’s reflexion principle to the mappings \(\tau_3\) and \(\tau_4\) it follows that, there exists a positive constant \(C_t'(K)\) such that

\[ \frac{1}{C_t'(K)} \leq |\nabla \hat{w}_t(z)| \leq C_t'(K), \quad z \in B_t(K), \]

where \(B_t(K) \subset B\) is a neighborhood of \(s\). Since \(\tau_1\) and \(\tau_2\) are diffeomorphisms, it follows that, there exists a constant \(C_t''(K)\) such that

\[ \frac{1}{C_t''(K)} \leq |\nabla w(z)| \leq C_t''(K), \quad z \in \Sigma_t(K), \]

where \(\Sigma_t(K) \subset \Sigma_1\) is a neighborhood of \(t\). Since \(\partial \Sigma_1\) is compact, there exists a positive constant \(C_t''(K)\) such that

\[ \frac{1}{C_t''(K)} \leq |\nabla w(z)| \leq C_t''(K), \quad z \in \Sigma(K), \]

where \(\Sigma(K)\) is a neighborhood of \(\partial \Sigma_1\). Finally we conclude that, there exists a positive constant \(C''(K)\) such that

\[ \frac{1}{C''(K)} \leq |\nabla w(z)| \leq C''(K), \quad z \in \Sigma_1. \]

The conclusion follows from the relations

\[ |\nabla w^{-1}(w(t))| = \frac{1}{l(\nabla w(t)^2)}, \]
the main value theorem and the fact that the surfaces are quasi-convex.

(iv) The general case. Let $\gamma$ be one of the boundary components of $\Sigma_1$. Then $\delta = w(\gamma)$ is a boundary component of $\Sigma_2$. Assume that $\gamma' \subset \Sigma_1 \setminus \gamma$ is a $C^{2,\alpha}$ Jordan curve homotopic to $\gamma$. Then $\delta' = w(\gamma') \in C^{2,\alpha}$ is homotopic to $\delta$. Let $A \subset \Sigma_1$ be the annulus generated by $\gamma$ and $\gamma'$. Applying the case (iii) to the mapping $w : A \to w(A)$ we obtain the desired conclusion.

Remark 3.3. Let $\lambda$ be the hyperbolic metric defined in (1.5). In [43, Theorem 13] is proved that, a $\lambda$-harmonic self-mapping of the unit disk is q.c. if and only if the function

$$\Psi = \frac{(1 - |z|^2)^2 w_z w_{\bar{z}}}{(1 - |w(z)|^2)^2}$$

is bounded. Moreover, concerning the hyperbolic metric, Wan showed that if $w$ is $k$-q.c. $\lambda$ harmonic, then it is a hyperbolic bi-Lipschitz self-mapping of the unit disk. See also [9].

The previous method gives a short proof of the theorem, that a q.c. harmonic mapping of the hyperbolic disk onto itself is bi-Lipschitz (one direction of Wan’s theorem).

To do so, denote by $e(w)$ the hyperbolic energy of a q.c. harmonic mapping of the unit disk onto itself:

$$e(w) = \frac{(1 - |z|^2)^2}{(1 - |w(z)|^2)^2} (|w_z|^2 + |w_{\bar{z}}|^2).$$

Assume there exists a sequence $(z_n)$ such that $e(w)(z_n) \to \infty$, or $e(w)(z_n) \to 0$, as $n \to \infty$. Take $w_n = p_n(w(q_n(z)))$, where $p_n$ and $q_n$ are Möbius transformations of the unit disk onto itself satisfying the conditions $p_n(w(z_n)) = 0$ and $q_n(0) = z_n$. Then, $w_n(0) = 0$ and up to some subsequence, $w_n \to w_0$ where $w_0$ is quasiconformal and harmonic. By [39] $\nabla w_0(0) \neq 0$.

But here we have

$$2|\nabla w_0(0)|^2 = \lim_{n \to \infty} 2|\nabla w_n(0)|^2 = \lim_{n \to \infty} e(w)(z_n) = \infty \text{ or } = 0.$$

This is a contradiction. Therefore there exists a constant $C \geq 1$ such that

$$\frac{1}{C} \lambda(z)|dz| \leq w^*(\lambda(z)|dz|) \leq C \lambda(z)|dz|$$

as desired.
3.1. **Open problems.** a) It is not known by the author if the mapping $w_0$ produced in Corollary 1.4 is bi-Lipschitz with respect to the Euclidean metric or what is a bit more, whether $\rho_0$ is an approximate analytic metric. This problem is open even for $\rho$ being a Euclidean metric. Also it is an interesting question if two sequences $z_n$ and $z'_n$ converges to the same boundary point $z_0$, do they induce the same harmonic mapping $w_0$ or at least the same metric $\rho_0$.

b) The Gauss curvature of a metric $\rho$ is given by

$$K = -\frac{\Delta \log \rho}{\rho^2}.$$  

Thus the Gauss curvature is positive if and only if

$$\Delta \rho(z) \leq |\nabla \rho(z)|^2, \quad z \in \mathbb{U}. \tag{3.4}$$

Heinz-Bersnetin theorem ([32]) states that: if

$$|\Delta \rho(z)| \leq |\nabla \rho(z)|^2, \quad z \in \mathbb{U} \tag{3.5}$$

and $\rho \in C^{1,\alpha}(S^1)$ then $|\nabla \rho|$ is bounded. Therefore $\rho$ is an approximate analytic metric.

Under these conditions on the metric $\rho$ the main theorem is true. The question arises whether the condition (3.5) can be replaced by (3.4).

**Acknowledgment.** I thank professor Miodrag Mateljević who showed interest in this paper, and suggested the statement of Theorem 1.4.

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